Project 6
The isometric stochastic integral

From my Mac’s dictionary:

iso·met·ric |i sooˈmɛtrɪk|
adjective
1 of or having equal dimensions.
2 Physiology of, relating to, or denoting muscular action in which tension is developed without contraction of the muscle.
3 (in technical or architectural drawing) incorporating a method of showing projection or perspective in which the three principal dimensions are represented by three axes 120° apart.
4 Mathematics (of a transformation) without change of shape or size.

DERIVATIVES
iso·met·ri·cal·ly -ɪkə(ˈlɛlɪ)| ɪˈsʊʊˌmɛtrɪk(ə)lɪ| ˌɪzəʊˈmetrɪk(ə)lɪ| adverb
iso·me·try |ɪˌsæmɪtrɪ| ərˈsæmətrɪ| ˈɪsəmətrɪ| noun (in sense 4).

ORIGIN mid 19th cent.: from Greek isometria ‘equality of measure’ (from isos ‘equal’ + -metria ‘measuring’). + -ic.

Lect 11, Monday 15 Feb

6.1 Notation and facts

All the processes to be considered in this project will live on a fixed complete probability space \((\Omega, \mathcal{F}, P)\), which is equipped with a standard filtration \(\{\mathcal{F}_t : 0 \leq t \leq 1\}\). As usual, abbreviate \(P(\cdots | \mathcal{F}_s)\) to \(P_s(\cdots)\).

- Define \(\mathcal{S} := \Omega \times (0, 1]\) and \(\overline{\mathcal{S}} := \Omega \times [0, 1].\) In the definition of the stochastic integral \(\int H \, dM\), the predictable process \(H\) will be defined on \(\mathcal{S}\) and the martingale \(M\) will be defined on \(\overline{\mathcal{S}}\).

- Write \(\mathcal{M}^2\), or \(\mathcal{M}^2[0, 1]\) if there is any ambiguity about the index set, for the vector space of all square integrable, cadlag martingales, that is, all martingales \(\{(M_t, \mathcal{F}_t) : 0 \leq t \leq 1\}\) with cadlag sample paths for which \(\mathbb{P}|M|^2 < \infty\). By Doob’s inequality (cf. Pollard 2001, Problem 6.9)

\[
1 < \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq t \leq 1} |M_t|^2 \right)^{1/2}
\]

for each \(M\) in \(\mathcal{M}^2\). Control of a martingale at time \(t = 1\) gives control over the whole index set \([0, 1]\). Define \(\mathcal{M}^2_0 = \{M \in \mathcal{M}^2[0, 1] : M_0 \equiv 0\}\).
• Two processes \( \{ X(\omega, t) : 0 \leq t \leq 1 \} \) and \( \{ Y(\omega, t) : 0 \leq t \leq 1 \} \) are said to be \( \mathbb{P} \)-indistinguishable if there exists a single \( \mathbb{P} \)-negligible set \( N \) such that \( X(\omega, t) = Y(\omega, t) \) for all \( (\omega, t) \in (\Omega \setminus N) \times [0,1] \).

• Write \( \mathcal{H}_{\text{simple}} \) for the set of all simple processes of the form

\[
H(t, \omega) = \sum_{i=0}^{N} h_i(\omega) \{ t_i < t \leq t_{i+1} \}
\]

for some finite grid \( 0 = t_0 < t_1 < \cdots < t_{N+1} = 1 \) and bounded, \( \mathcal{F}(t_i) \)-measurable random variables \( h_i \). As defined, \( H(\omega, 0) \equiv 0 \). Indeed, if I am interpreting Dellacherie and Meyer (1978, IV.61(b)) correctly, it is better to think of such an \( H \) as being defined on the set \( \mathcal{G} := \Omega \times (0,1] \).

Remark. Some authors call members of \( \mathcal{H}_{\text{simple}} \) elementary processes; others reserve that name for the situation where the \( t_i \) are replaced by stopping times. Dellacherie and Meyer (1982, §8.1) adopted the opposite convention.

• Write \( \mathcal{P} \) for the predictable sigma-field on \( \mathcal{G} \) generated by \( \mathcal{H}_{\text{simple}} \). A process is said to be predictable if it is \( \mathcal{P} \)-measurable. As you will see in Section 4, \( \mathcal{P} \) has a few other useful generating classes, such as the collection of all subsets of \( \mathcal{G} \) of the form \( F \times (a,b] \) for \( 0 \leq a < b \leq 1 \) and \( F \in \mathcal{F}_a \).

The main task in this project is to prove the existence of the isometric stochastic integral with respect to a martingale in \( \mathcal{M}^2 \). The isometry involves the Doléans measure associated with the submartingale \( M^2 \).

Lemma. (existence of the Doléans measure) For each \( M \) in \( \mathcal{M}^2[0,1] \) there exists a unique finite measure \( \mu = \mu_M \) on \( \mathcal{P} \) for which

\[
\mu F \times (a,b] = \mathbb{P} F(M_b - M_a)^2
\]

for each \( 0 \leq a < b \leq 1 \) and each \( F \in \mathcal{F}_a \).

• The martingale \( M \) enters the definition of \( \mu_M \) only through its increments. We lose no generality in considering only \( M \) in \( \mathcal{M}_0^2 \).

• Write \( \mathcal{H}^2(\mu_M) \), or just \( \mathcal{H}^2 \), for the set of all predictable processes \( H \) on \( \mathcal{G} \) for which \( \mu H^2 \leq \infty \).

Appendix C proves existence of the Doléans measure for a large class of submartingales (which includes \( M^2 \) for each \( M \) in \( \mathcal{M}^2 \)).
6.2 Stochastic integral for simple processes

If $H \in \mathcal{H}_{\text{simple}}$, as in <2>, and $M \in \mathcal{M}^2$, define

$$\int_{(0,1]} H \, dM := \sum_{i=0}^{N} h_i(\omega) \left( M(\omega, t_{i+1}) - M(\omega, t_i) \right).$$

Here I follow Rogers and Williams (1987, page 2) in excluding the lower endpoint from the range of integration. Dellacherie and Meyer (1982, §8.1) added an extra contribution from a possible jump in $M$ at 0. With the $(0,1]$ interpretation, the definition depends only on the increments of $M$; with no loss of generality, we may therefore assume $M_0 \equiv 0$.

A similar awkwardness arises in defining $\int_{0}^{t} H \, dM$ if $M$ has a jump at $t$. The notation does not distinguish between the integral over $(0,t)$ and the integral over $(0,t]$. I will use instead the Strasbourg notation $H \cdot M_1$ for $\int_{(0,1]} H \, dM$, with $H$ multiplied by an explicit indicator function to modify the range of integration. For example, $\int_{0}^{t} H \, dM$ is obtained from <4> by substituting $H(\omega, s)\{0 < s \leq t\}$ for $H$. Thus

$$H \cdot M_t := (H(\omega, s)\{0 < s \leq t\}) \cdot M_1$$

$$= \sum_{i=0}^{N} h_i(\omega) \left( M(\omega, t \wedge t_{i+1}) - M(\omega, t \wedge t_i) \right).$$

- You should check that $\mathbb{P}(H \cdot M_1) = H \cdot M_t$ almost surely, so that $H \cdot M$ is a martingale (with cadlag paths).

The next theorem states the basic facts about the isometric stochastic integral. It will be proved in the remaining sections of this project.

**Theorem.** For each $M$ in $\mathcal{M}^2_0[0,1]$ there exists a linear map $H \mapsto H \cdot M$ from $\mathcal{H}^2(\mu_M)$ onto a closed (for $\|\cdot\|_M$) subspace of $\mathcal{M}^2_0$ for which

(i) if $H = \sum_{i=0}^{N} h_i(\omega)\{t_i < t \leq t_{i+1}\} \in \mathcal{H}_{\text{simple}}$ then

$$H \cdot M_t = \sum_{i=0}^{N} h_i(\omega) \left( M(\omega, t_{i+1} \wedge t) - M(\omega, t_i \wedge t) \right) \quad \text{for } 0 \leq t \leq 1$$

(ii) $\mathbb{P}(H \cdot M_1)^2 = \mu_M H^2$ for each $H \in \mathcal{H}^2(\mu_M)$.

That is, $H \mapsto H \cdot M_1$ is an isometry from $\mathcal{H}^2(\mu_M)$ onto a closed subspace of $L^2(\Omega, \mathcal{F}_1, \mathbb{P})$. 
Increasing processes as measures

I will start the extension of $H \bullet M$ to more general $H$ processes by assuming slightly more than the existence of the Doléans measure. Instead I assume the existence of a cadlag, adapted process $A$ with increasing sample paths for which $N_t := M_t^2 - A_t$ is a martingale. For example, for Brownian motion, $A_t(\omega) \equiv t$. As you will see in Section 6, we don’t actually need the process $A$ for the construction of the stochastic integral if we restrict ourselves to predictable integrands.

**Remark.** If $A$ is a predictable process, the representation $M^2 = N + A$ is called the Doob-Meyer decomposition. The proof of existence of such an $A$ for each $M$ in $\mathcal{M}_2([0,1])$ involves a lot of work. The construction of $\mu_M$ on $\mathcal{F}$ is the much easier first step. There is a procedure (the dual predictable projection) for extending $\mu_M$ to a “predictable measure” on $\mathcal{B}(0,1) \otimes \mathcal{F}_1$. A disintegration of this new measure then defines the process $A$.

Without loss of generality, $A_0 = 0$. Identify $A(\cdot, \omega)$ with the measure $\mu_\omega$ on $\mathcal{B}(0,1]$ for which

$$\mu_\omega(0, t] = A(t, \omega) \quad \text{for } 0 < t \leq 1.$$ 

See Pollard (2001, Section 2.9) for details of how to build $\mu_\omega$ as an image measure (the quantile transformation). Construct a measure $\mu$ on $\mathcal{B}(0,1] \otimes \mathcal{F}$ by

$$\mu g(t, \omega) = P^\omega \mu_\omega g(\omega, t)$$

at least for nonnegative, $\mathcal{B}(0,1] \otimes \mathcal{F}$-measurable functions $g$. (In the notation of Pollard 2001, Section 4.3, the measure $\mu$ equals $P \otimes \{\mu_\omega : \omega \in \Omega\}$.) Notice that $\mu(0, 1] \times \Omega = P A_1 < \infty$.

(i) For Brownian motion, show that $\mu = P \otimes m$ with $m = \text{Lebesgue measure}$ on $\mathcal{B}(0,1]$.

(ii) For fixed $0 \leq a < b \leq 1$, define $\Delta N = N_b - N_a$, $\Delta M = M_b - M_a$, and $\Delta A = A_b - A_a$. Show that

$$0 = P_a \Delta N = P_a ((\Delta M)^2 - \Delta A) \quad \text{almost surely.}$$

(iii) At least for each bounded, $\mathcal{F}_a$-measurable random variable $h$, deduce that

$$P h(\omega)(\Delta M)^2 = P h(\omega) \Delta A = P^\omega (h(\omega) \mu_\omega^a \{a < t \leq b\})$$

$$= \mu h(\omega) \{a < t \leq b\}. \quad \text{<7>}$$
\section*{\S 6.4 The predictable sigma-field}

I defined \( \mathcal{P} \) to be \( \sigma(\mathcal{H}_{\text{simple}}) \), the smallest sigma-field on \( \mathcal{S} \) for which each member of \( \mathcal{H}_{\text{simple}} \) is \( \mathcal{P} \setminus \mathcal{B}(\mathbb{R}) \)-measurable. You should check that \( \mathcal{P} \) is also generated by the following collections of sets or processes.

(a) the collection \( \mathcal{E} \) of all subsets of \( \mathcal{S} \) of the form \( F \times (a, b) \) for \( 0 \leq a < b \leq 1 \) and \( F \in \mathcal{F}_\alpha \)

(b) the set \( \mathcal{H}_{\text{left}} \) of all adapted process \( L \) on \( \mathcal{S} \) with sample paths that are left-continuous at each point of \( (0, 1] \).

(c) the set \( \mathcal{C} \) of restrictions to \( \mathcal{S} \) of adapted processes on \( \Omega \times [0, 1] \) with continuous sample paths

(d) the set of all stochastic intervals

\[ ([0, \tau]) := \{(\omega, t) \in \mathcal{S} : 0 < t \leq \tau(\omega)\} \]

where \( \tau \) ranges over the set \( \mathcal{T} \) of all stopping times for the filtration.

\textbf{Remark.} Note that \( ([0, \tau]) \) is unchanged if we replace \( \tau \) by \( \tau \wedge 1 \); the point \( (\omega, \infty) \) never belongs to the stochastic interval, even if \( \tau(\omega) = +\infty \). D&M write \([0, \tau] \) for \( ([0, \tau]) \).

(e) the set of all processes of the form

\[ (\sigma, \tau]h(\omega) = h(\omega)\{(\omega, t) \in \mathcal{S} : \sigma(\omega) < t \leq \tau(\omega)\} \]

where \( \sigma \) and \( \tau \) are stopping times, with \( \sigma \leq \tau \), and \( h \) is \( \mathcal{F}_\sigma \)-measurable
(f) the vector space $\mathcal{H}_{\text{BddLip}}$ of restrictions to $\mathcal{G}$ of adapted processes $H$ on $[0, 1] \times \Omega$ for which there exists a finite constant $C_H$ such that $|H(\omega, t)| \leq C_H$ and $|H(\omega, s) - H(\omega, t)| \leq C_H |t - s|$ for all $s, t,$ and $\omega$.

**Remark.** In Appendix C, the Doléans measure will be defined as a linear functional on $\mathcal{H}_{\text{BddLip}}$.

**Proof**

(i) For (a): Note that each $\mathcal{E}$-set is in $\mathcal{H}_{\text{simple}}$. For $h(\omega)\{a < t \leq b\}$ with $h$ bounded and $\mathcal{F}_a$-measurable, approximate $h$ by simple functions.

(ii) For (b): All $\mathcal{H}_{\text{simple}}$ processes belong to $\mathcal{H}_{\text{left}}$. Express an $H$ in $\mathcal{H}_{\text{left}}$ as a pointwise limit of $\mathcal{H}_{\text{simple}}$ processes,

$$H_n(\omega, t) := \sum_{i=2}^{2^n} H(\omega, t_{n,i-1}) \{ t_{n,i-1} < t \leq t_{n,i} \}$$

where $t_{n,i} := i/2^n$

(iii) For (d): Without loss of generality suppose $\tau$ is a stopping time taking values in $[0, 1]$. Let $\tau_n$ be the stopping time obtained by rounding $\tau$ up to the next integer multiple of $2^{-n}$, that is, $\tau_n(\omega) := 2^{-n} \lceil 2^n \tau(\omega) \rceil$. Show that

$$((0, \tau_n]) = \sum_{i=1}^{2^n} \{ t_{i-1} < t \leq t_i \} \{ \tau(\omega) > t_{i-1} \} \in \mathcal{H}_{\text{simple}}$$

and that $\cap_{n \in \mathbb{N}} ((0, \tau_n]) = ((0, \tau])$. Also, if $F \times (a, b) \in \mathcal{E}$, show that $\sigma(\omega) := a \{ \omega \in F \} + \{ \omega \in F^c \}$ and $\tau(\omega) := b \{ \omega \in F \} + \{ \omega \in F^c \}$ are stopping times for which $F \times (a, b) = ((\sigma, \tau])$.

(iv) For (e): Approximate $\sigma$ and $\tau$ as for (d).

(v) For (f): If $F \times (a, b) \in \mathcal{E}$ define

$$H_n(\omega, t) := \{ \omega \in F \} \left( 1 - n(t - a - n^{-1})^+ \right)^+ \in \mathcal{H}_{\text{BddLip}}.$$

Then $H_n \to F \times (a, 1]$ pointwise as $n \to \infty$. Argue similarly for $F \times (b, 1]$.

### 6.5 Extension by isometry

Think of $\mathcal{H}_{\text{simple}}$ as a subspace of $L^2(\mu) := L^2(\mathcal{G}, \mathcal{B}(0, 1] \otimes \mathcal{F}_1, \mu)$. Then Lemma <8> shows that $H \mapsto H \circ M_1$ is an isometry from a subspace of $L^2(\mu)$ to $L^2(\mathbb{P}) := L^2(\Omega, \mathcal{F}_1, \mathbb{P})$. It extends to an isometry from $\mathcal{H}_{\text{simple}}$,
the $\mathcal{L}^2(\mu)$ closure of $\mathcal{H}_{\text{simple}}$ in $\mathcal{L}^2(\mu)$, into $\mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$. To avoid confusion between norms, write $\|H\|_{\mathcal{L}^2(\mu)}$ or $\|H\|_\mu$ for the $\mathcal{L}^2(\mu)$ norm and $\|X\|_{\mathcal{L}^2(\mathbb{P})}$ or $\|X\|_\mathbb{P}$ for the $\mathcal{L}^2(\mathbb{P})$ norm of a random variable $X$.

(i) For each $G \in \mathcal{F}_{\text{simple}}$ there exists a sequence $\{H_n\}$ in $\mathcal{H}_{\text{simple}}$ for which $\|G - H_n\|_{\mathcal{L}^2(\mu)} \to 0$. Show that $\{H_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$. Deduce, via Lemma 8, that $\{H_n \cdot M_1\}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbb{P})$, which therefore converges to some $Z$ in $\mathcal{L}^2(\mathbb{P})$.

(ii) If $\{K_n\}$ is another sequence in $\mathcal{H}_{\text{simple}}$ for which $\|G - K_n\|_{\mathcal{L}^2(\mu)} \to 0$, show that $\|H_n \cdot M_1 - K_n \cdot M_1\|_{\mathcal{L}^2(\mathbb{P})} \to 0$. Deduce that $K_n \cdot M_1$ also converges to $Z$ in $\mathcal{L}^2(\mathbb{P})$ norm.

(iii) Define (up to an almost sure equivalence) $G \cdot M_1 = Z$.

(iv) For each $t$ in $[0, 1]$, argue similarly that $G \cdot M_t$ could be defined as an $\mathcal{L}^2(\mathbb{P})$ limit of $H_n \cdot M_t$.

(v) Show that $\mathbb{P}_t(G \cdot M_1) = G_t$ almost surely. Choose a cadlag version of the martingale $\{(G \cdot M_t, \mathcal{F}_t) : 0 \leq t \leq 1\}$. Show that $G \mapsto G \cdot M$ is linear (up to some sort of almost sure equivalence).

(vi) Show that $\|G \cdot M\|_\mathcal{M} := \|G \cdot M_1\|_{\mathcal{L}^2(\mathbb{P})} = \|G\|_{\mathcal{L}^2(\mu)}$.

(vii) Suppose $\{G_t\}$ is a sequence in $\mathcal{F}_{\text{simple}}$ with $\|G_t \cdot M_1 - W\|_{\mathcal{L}^2(\mathbb{P})} \to 0$ for some $W \in \mathcal{L}^2(\mathbb{P})$. Show that $\|G_t - G\|_{\mathcal{L}^2(\mu)} \to 0$ for some $G \in \mathcal{F}_{\text{simple}}$ and $W = G \cdot M_1$ almost surely.

Deduce that $\{G \cdot M : G \in \mathcal{F}_{\text{simple}}\}$ is a $\|\cdot\|_\mathcal{M}$ closed subspace of $\mathcal{M}_0^2$. [Strictly speaking, we should work with equivalence classes of indistinguishable processes.]

(viii) Show that the $H \cdot M$ described in Theorem 6 is unique up to indistinguishability.
Example. Suppose $\sigma$ and $\tau$ are stopping time taking values in $[0, 1]$, with $\sigma \leq \tau$. Suppose $h$ is a bounded, $\mathcal{F}_\sigma$-measurable random variable. Define $\sigma_n$ to be the stopping time obtained by rounding $\sigma$ up to the next integer multiple of $2^{-n}$ and define $\tau_n$ similarly.

- Show that $\mu h(\omega) \left( ((\sigma_n, \tau_n]) - ((\sigma, \tau]] \right)^2 \to 0$.
- Deduce that $(h(\omega)((\sigma, \tau]]) \bullet M_t = h(\omega) (M_{t\wedge \tau} - M(\sigma \wedge t))$.

□

6.6 Predictable integrands

How large is $\mathcal{H}_{\text{simple}}$?

(i) Invoke a $\lambda$-space argument to show that $\mathcal{H}_{\text{simple}}$ contains all bounded, $\mathcal{P}$-measurable processes.

(ii) In general, if $G \in \mathcal{H}_{\text{simple}}$ then $\|G - H_n\|_{\mathcal{L}_2(\mu)} \to 0$ for some sequence $\{H_n\}$ in $\mathcal{H}_{\text{simple}}$. There exists a subsequence along which $H_n \to G$ a.e. $[\mu]$. As each $H_n$ is $\mathcal{P}$-measurable, there must exists some $\mathcal{P}$-measurable $G^*$ with $G^* = G$ a.e. $[\mu]$.

(iii) For Brownian motion, that $\mathcal{H}_{\text{simple}}$ contains all the $\mathcal{F}_1 \otimes \mathcal{B}(0, 1]$-measurable, adapted processes that are square integrable for $\mathbb{P} \times \mathbb{m}$. (See Chung and Williams 1990, Theorem 3.7 or Problem [2] below.) We don’t lose much by restricting ourselves to $\mathcal{P}$-measurable integrands.

6.7 Problems

[1] Suppose $M \in \mathcal{M}^2[0, 1]$ has continuous sample paths.

(i) For each $H$ in $\mathcal{H}_{\text{simple}}$, show that $H \bullet M$ has continuous sample paths.

(ii) Suppose $\{H_n : n \in \mathbb{N}\} \subseteq \mathcal{H}_{\text{simple}}$ and $\mu |H_n - H|^2 < 2^{-n}$. Use Doob’s maximal inequality to show that

$$\sum_n \mathbb{P} \sup_{0 \leq t \leq 1} |H_n \bullet M_t - H \bullet M_t| < \infty$$

(iii) Deduce that there is a version of $H \bullet M$ with continuous sample paths.
Suppose $\mu = m \otimes \mathbb{P}$, defined on $\mathcal{B}(0,1] \otimes \mathcal{F}$. Let $\{X_t : 0 \leq t \leq 1\}$ be progressively measurable.

(i) Suppose $X$ is bounded, that is, $\sup_{t, \omega} |X(t, \omega)| < \infty$. Define

$$H_n(t, \omega) := n \int_{t-n^{-1}}^t X(s, \omega) \, ds$$

(How should you understand the definition when $t < n^{-1}$?) Show that $H_n$ is predictable and $\int_0^1 |H_n(t, \omega) - X(t, \omega)|^2 \, dt \to 0$ for each $\omega$.

(ii) Deduce that $\mu |H_n - X|^2 \to 0$.

(iii) Deduce that $X \in \mathcal{F}_{\text{simple}}$ if $\mu X^2 < \infty$.

References


