Project 6

The isometric stochastic integral

From my Mac's dictionary:

i•so•met•ric |ˌīsəˈmetrik|
adjective
1 of or having equal dimensions.
2 Physiology of, relating to, or denoting muscular action in which tension is developed without contraction of the muscle.
3 (in technical or architectural drawing) incorporating a method of showing projection or perspective in which the three principal dimensions are represented by three axes 120° apart.
4 Mathematics (of a transformation) without change of shape or size.

DERIVATIVES
i•so•met•ri•cal•ly |-ik(ə)lē| |ˈaɪsoʊˈmɛtrək(ə)li| |ˈaɪzəˈmɛtrək(ə)li| adverb
i•som•e•try |īˈsämitrē| |aɪˈsɑmətri| |ʌɪˈsɒmɪtri| noun (in sense 4).

ORIGIN mid 19th cent.: from Greek ***isometria* 'equality of measure'** (from ***isos* 'equal'** + **-*metria* 'measuring'**) + **-IC** .

Lect 11, Monday 15 Feb

6.1 Notation and facts

All the processes to be considered in this project will live on a fixed complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is equipped with a standard filtration $\{\mathcal{F}_t : 0 \le t \le 1\}$. As usual, abbreviate $\mathbb{P}(\cdots \mid \mathcal{F}_s)$ to $\mathbb{P}_s(\dots)$.

- Define $\mathfrak{S} := \Omega \times (0,1]$ and $\overline{\mathfrak{S}} := \Omega \times [0,1]$. In the definition of the stochastic integral $\int H\,dM$, the ***predictable*** process H will be defined on $\mathfrak{S}$ and the martingale M will be defined on $\overline{\mathfrak{S}}$.

- Write $\mathcal{M}^2$, or $\mathcal{M}^2[0,1]$ if there is any ambiguity about the index set, for the vector space of all square integrable, cadlag martingales, that is, all martingales $\{(M_t, \mathcal{F}_t) : 0 \le t \le 1\}$ with cadlag sample paths for which $\mathbb{P}M_1^2 < \infty$. By Doob's inequality (cf. Pollard 2001, Problem 6.9)

<1> $$\|M\|_{\mathcal{M}} := \|M_1\|_2 \ge \tfrac{1}{2}\left(\mathbb{P}\sup_{0\le t\le 1} M_t^2\right)^{1/2}$$

  for each M in $\mathcal{M}^2$. Control of a martingale at time $t = 1$ gives control over the whole index set $[0,1]$. Define $\mathcal{M}_0^2 = \{M \in \mathcal{M}^2[0,1] : M_0 \equiv 0\}$.

- Two processes $\{X(\omega, t) : 0 \le t \le 1\}$ and $\{Y(\omega, t) : 0 \le t \le 1\}$ are said to be $\mathbb{P}$-***indistinguishable*** if there exists a single $\mathbb{P}$-negligible set $\mathcal{N}$ such that $X(\omega, t) = Y(\omega, t)$ for all $(\omega, t) \in (\Omega \backslash \mathcal{N}) \times [0, 1]$.

- Write $\mathcal{H}_{\text{simple}}$ for the set of all ***simple processes*** of the form

<2> $$H(t, \omega) = \sum\nolimits_{i=0}^{N} h_i(\omega)\{t_i < t \le t_{i+1}\} \qquad \text{for } 0 \le t \le 1$$

  for some finite grid $0 = t_0 < t_1 < \cdots < t_{N+1} = 1$ and bounded, $\mathcal{F}(t_i)$-measurable random variables h_i. As defined, $H(\omega, 0) \equiv 0$. Indeed, if I am interpreting Dellacherie and Meyer (1978, IV.61(b)) correctly, it is better to think of such an H as being defined on the set $\mathfrak{S} := \Omega \times (0, 1]$.

  Remark. Some authors call members of $\mathcal{H}_{\text{simple}}$ ***elementary processes***; others reserve that name for the situation where the t_i are replaced by stopping times. Dellacherie and Meyer (1982, §8.1) adopted the opposite convention.

- Write $\mathcal{P}$ for the ***predictable*** sigma-field on $\mathfrak{S}$ generated by $\mathcal{H}_{\text{simple}}$. A process is said to be predictable if it is $\mathcal{P}$-measurable. As you will see in Section 4, $\mathcal{P}$ has a few other useful generating classes, such as the collection of all subsets of $\mathfrak{S}$ of the form $F \times (a, b]$ for $0 \le a < b \le 1$ and $F \in \mathcal{F}_a$.

The main task in this project is to prove the existence of the isometric stochastic integral with respect to a martingale in $\mathcal{M}^2$. The isometry involves the ***Doléans measure*** associated with the submartingale M^2.

<3> **Lemma.** *(existence of the Doléans measure) For each M in $\mathcal{M}^2[0, 1]$ there exists a unique finite measure $\mu = \mu_M$ on $\mathcal{P}$ for which*

$$\mu F \times (a, b] = \mathbb{P} F (M_b - M_a)^2$$

for each $0 \le a < b \le 1$ and each $F \in \mathcal{F}_a$.

- The martingale M enters the definition of μ_M only through its increments. We lose no generality in considering only M in $\mathcal{M}_0^2$.

- Write $\mathcal{H}^2(\mu_M)$, or just $\mathcal{H}^2$, for the set of all predictable processes H on $\mathfrak{S}$ for which $\mu H^2 < \infty$.

Appendix C proves existence of the Doléans measure for a large class of submartingales (which includes M^2 for each M in $\mathcal{M}^2$).

6.2 Stochastic integral for simple processes

If $H \in \mathcal{H}_{\text{simple}}$, as in <2>, and $M \in \mathcal{M}^2$, define

<4>
$$\int_{(0,1]} H\,dM := \sum\nolimits_{i=0}^{N} h_i(\omega)\left(M(\omega,t_{i+1}) - M(\omega,t_i)\right).$$

Here I follow Rogers and Williams (1987, page 2) in excluding the lower endpoint from the range of integration. Dellacherie and Meyer (1982, §8.1) added an extra contribution from a possible jump in M at 0. With the $(0,1]$ interpretation, the definition depends only on the increments of M; with no loss of generality, we may therefore assume $M_0 \equiv 0$.

A similar awkwardness arises in defining $\int_0^t H\,dM$ if M has a jump at t. The notation does not distinguish between the integral over $(0,t)$ and the integral over $(0,t]$. I will use instead the Strasbourg notation $H \bullet M_1$ for $\int_{(0,1]} H\,dM$, with H multiplied by an explicit indicator function to modify the range of integration. For example, $\int_0^t H\,dM$ is obtained from <4> by substituting $H(s,\omega)\{0 < s \le t\}$ for H. Thus

<5>
$$\begin{aligned} H \bullet M_t &:= (H(\omega,s)\{0 < s \le t\}) \bullet M_1 \\ &= \sum\nolimits_{i=0}^{N} h_i(\omega)\left(M(\omega,t\wedge t_{i+1}) - M(\omega,t\wedge t_i)\right). \end{aligned}$$

- You should check that $\mathbb{P}_t H \bullet M_1 = H \bullet M_t$ almost surely, so that $H \bullet M$ is a martingale (with cadlag paths).

The next theorem states the basic facts about the isometric stochastic integral. It will be proved in the remaining sections of this project.

<6> **Theorem.** *For each M in $\mathcal{M}_0^2[0,1]$ there exists a linear map $H \mapsto H \bullet M$ from $\mathcal{H}^2(\mu_M)$ onto a closed (for $\|\cdot\|_{\mathcal{M}}$) subspace of $\mathcal{M}_0^2$ for which*

(i) if $H = \sum_{i=0}^{N} h_i(\omega)\{t_i < t \le t_{i+1}\} \in \mathcal{H}_{\text{simple}}$ then

$$H \bullet M_t = \sum\nolimits_{i=0}^{N} h_i(\omega)\left(M(\omega,t_{i+1}\wedge t) - M(\omega,t_i\wedge t)\right) \qquad \text{for } 0 \le t \le 1$$

(ii) $\mathbb{P}(H \bullet M_1)^2 = \mu_M H^2$ for each $H \in \mathcal{H}^2(\mu_M)$.

That is, $H \mapsto H \bullet M_1$ is an isometry from $\mathcal{H}^2(\mu_M)$ onto a closed subspace of $\mathcal{L}^2(\Omega,\mathcal{F}_1,\mathbb{P})$.

6.3 Increasing processes as measures

I will start the extension of $H \bullet M$ to more general H processes by assuming slightly more than the existence of the Doléans measure. Instead I assume the existence of a cadlag, adapted process A with increasing sample paths for which $N_t := M_t^2 - A_t$ is a martingale. For example, for Brownian motion, $A_t(\omega) \equiv t$. As you will see in Section 6, we don't actually need the process A for the construction of the stochastic integral if we restrict ourselves to predictable integrands.

Remark. If A is a predictable process, the representation $M^2 = N + A$ is called the ***Doob-Meyer*** decomposition. The proof of existence of such an A for each M in $\mathcal{M}_0^2[0,1]$ involves a lot of work. The construction of μ_M on $\mathcal{P}$ is the much easier first step. There is a procedure (the dual predictable projection) for extending μ_M to a "predictable measure" on $\mathcal{B}(0,1] \otimes \mathcal{F}_1$. A disintegration of this new measure then defines the process A.

Without loss of generality, $A_0 = 0$. Identify $A(\cdot, \omega)$ with the measure μ_ω on $\mathcal{B}(0,1]$ for which

$$\mu_\omega(0,t] = A(t,\omega) \qquad \text{for } 0 < t \le 1.$$

See Pollard (2001, Section 2.9) for details of how to build μ_ω as an image measure (the quantile transformation). Construct a measure μ on $\mathcal{B}(0,1] \otimes \mathcal{F}$ by

$$\mu g(t,\omega) = \mathbb{P}^\omega \mu_\omega^t g(\omega, t)$$

at least for nonnegative, $\mathcal{B}(0,1] \otimes \mathcal{F}$-measurable functions g. (In the notation of Pollard 2001, Section 4.3, the measure μ equals $\mathbb{P} \otimes \{\mu_\omega : \omega \in \Omega\}$.) Notice that $\mu(0,1] \times \Omega = \mathbb{P}A_1 < \infty$.

(i) For Brownian motion, show that $\mu = \mathbb{P} \otimes \mathfrak{m}$ with $\mathfrak{m}$ = Lebesgue measure on $\mathcal{B}(0,1]$.

(ii) For fixed $0 \le a < b \le 1$, define $\Delta N = N_b - N_a$, $\Delta M = M_b - M_a$, and $\Delta A = A_b - A_a$. Show that

$$0 = \mathbb{P}_a \Delta N = \mathbb{P}_a \left((\Delta M)^2 - \Delta A \right) \qquad \text{almost surely.}$$

(iii) At least for each bounded, $\mathcal{F}_a$-measurable random variable h, deduce that

$$\begin{aligned} \mathbb{P}h(\omega)(\Delta M)^2 &= \mathbb{P}h(\omega)\Delta A = \mathbb{P}^\omega \left(h(\omega) \mu_\omega^t \{a < t \le b\} \right) \\ &= \mu h(\omega)\{a < t \le b\}. \end{aligned} \qquad \text{<7>}$$

<8> **Lemma.** $\mathbb{P}\left(H \bullet M_1\right)^2 = \mu H^2$ *for each* $H \in \mathcal{H}_{\text{simple}}$*,*

PROOF Expand the left-hand side of the asserted inequality as

$$\sum\nolimits_i \mathbb{P}h_i^2(\Delta_i M)^2 + 2\sum\nolimits_{i<j} \mathbb{P}h_i h_j \Delta_i M \Delta_j M \qquad \text{where } \Delta_i M := M(t_{i+1} - M(t_i).$$

Use the fact that $\mathbb{P}(\Delta_j M \mid \mathcal{F}(t_{j-1})) = 0$ to kill all the cross-product terms. Use equality <7> to simplify the other contributions to

$$\mu^{s,\omega} \sum\nolimits_i h_i(\omega)^2\{t_i < s \le t_{i+1}\} = \mu H^2.$$

□

6.4 The predictable sigma-field

I defined $\mathcal{P}$ to be $\sigma(\mathcal{H}_{\text{simple}})$, the smallest sigma-field on $\mathfrak{S}$ for which each member of $\mathcal{H}_{\text{simple}}$ is $\mathcal{P}\backslash\mathcal{B}(\mathbb{R})$-measurable. You should check that $\mathcal{P}$ is also generated by the following collections of sets or processes.

(a) the collection $\mathcal{E}$ of all subsets of $\mathfrak{S}$ of the form $F \times (a, b]$ for $0 \le a < b \le 1$ and $F \in \mathcal{F}_a$

(b) the set $\mathcal{H}_{\text{left}}$ of all adapted process L on $\mathfrak{S}$ with sample paths that are left-continuous at each point of $(0, 1]$.

(c) the set $\mathbb{C}$ of restrictions to $\mathfrak{S}$ of adapted processes on $\Omega \times [0, 1]$ with continuous sample paths

(d) the set of all stochastic intervals

$$((0, \tau]] := \{(\omega, t) \in \mathfrak{S} : 0 < t \le \tau(\omega)\}$$

where τ ranges over the set $\mathcal{T}$ of all stopping times for the filtration.

> **Remark.** Note that $((0, \tau]]$ is unchanged if we replace τ by $\tau \wedge 1$; the point (ω, ∞) never belongs to the stochastic interval, even if $\tau(\omega) = +\infty$. D&M write $]]0, \tau]]$ for $((0, \tau]]$.

(e) the set of all processes of the form

$$((\sigma, \tau]]h(\omega) = h(\omega)\{(\omega, t) \in \mathfrak{S} : \sigma(\omega) < t \le \tau(\omega)\}$$

where σ and τ are stopping times, with $\sigma \le \tau$, and h is $\mathcal{F}_\sigma$-measurable

(f) the vector space $\mathcal{H}_{\text{BddLip}}$ of restrictions to $\mathfrak{S}$ of adapted processes H on $[0,1] \times \Omega$ for which there exists a finite constant C_H such that $|H(\omega,t)| \le C_H$ and $|H(\omega,s) - H(\omega,t)| \le C_H|t-s|$ for all s, t, and ω.

Remark. In Appendix C, the Doléans measure will be defined as a linear functional on $\mathcal{H}_{\text{BddLip}}$.

Proof

(i) For (a): Note that each $\mathcal{E}$-set is in $\mathcal{H}_{\text{simple}}$. For $h(\omega)\{a < t \le b\}$ with h bounded and $\mathcal{F}_a$-measurable, approximate h by simple functions.

(ii) For (b): All $\mathcal{H}_{\text{simple}}$ processes belong to $\mathcal{H}_{\text{left}}$. Express an H in $\mathcal{H}_{\text{left}}$ as a pointwise limit of $\mathcal{H}_{\text{simple}}$ processes,

$$H_n(\omega,t) := \sum\nolimits_{i=2}^{2^n} H(\omega, t_{n,i-1})\{t_{n,i-1} < t \le t_{n,i}\} \qquad \text{where } t_{n,i} := i/2^n$$

(iii) For (d): Without loss of generality suppose τ is a stopping time taking values in $[0,1]$. Let τ_n be the stopping time obtained by rounding τ up to the next integer multiple of 2^{-n}, that is, $\tau_n(\omega) := 2^{-n}\lceil 2^n\tau(\omega)\rceil$. Show that

$$((0,\tau_n]] = \sum\nolimits_{i=1}^{2^n} \{t_{i-1} < t \le t_i\}\{\tau(\omega) > t_{i-1}\} \in \mathcal{H}_{\text{simple}}$$

and that $\cap_{n\in\mathbb{N}} ((0,\tau_n]] = ((0,\tau]]$. Also, if $F \times (a,b] \in \mathcal{E}$, show that $\sigma(\omega) := a\{\omega \in F\} + \{\omega \in F^c\}$ and $\tau(\omega) := b\{\omega \in F\} + \{\omega \in F^c\}$ are stopping times for which $F \times (a,b] = ((\sigma,\tau]]$.

(iv) For (e): Approximate σ and τ as for (d).

(v) For (f): If $F \times (a,b] \in \mathcal{E}$ define

$$H_n(\omega,t) := \{\omega \in F\}\left(1 - n(t-a-n^{-1})^+\right)^+ \in \mathcal{H}_{\text{BddLip}}.$$

Then $H_n \to F \times (a,1]$ pointwise as $n \to \infty$. Argue similarly for $F \times (b,1]$.

6.5 Extension by isometry

Think of $\mathcal{H}_{\text{simple}}$ as a subspace of $\mathcal{L}^2(\mu) := \mathcal{L}^2(\mathfrak{S}, \mathcal{B}(0,1] \otimes \mathcal{F}_1, \mu)$. Then Lemma <8> shows that $H \mapsto H \bullet M_1$ is an isometry from a subspace of $\mathcal{L}^2(\mu)$ to $\mathcal{L}^2(\mathbb{P}) := \mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$. It extends to an isometry from $\overline{\mathcal{H}}_{\text{simple}}$, the $\mathcal{L}^2(\mu)$ closure of $\mathcal{H}_{\text{simple}}$ in $\mathcal{L}^2(\mu)$, into $\mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$. To avoid confusion between norms, write $\|H\|_{\mathcal{L}^2(\mu)}$ or $\|H\|_\mu$ for the $\mathcal{L}^2(\mu)$ norm and $\|X\|_{\mathcal{L}^2(\mathbb{P})}$ or $\|X\|_{\mathbb{P}}$ for the $\mathcal{L}^2(\mathbb{P})$ norm of a random variable X.

(i) For each $G \in \overline{\mathcal{H}}_{\text{simple}}$ there exists a sequence $\{H_n\}$ in $\mathcal{H}_{\text{simple}}$ for which $\|G - H_n\|_{\mathcal{L}^2(\mu)} \to 0$. Show that $\{H_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$. Deduce, via Lemma 8, that $\{H_n \bullet M_1\}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbb{P})$, which therefore converges to some Z in $\mathcal{L}^2(\mathbb{P})$.

(ii) If $\{K_n\}$ is another sequence in $\mathcal{H}_{\text{simple}}$ for which $\|G - K_n\|_{\mathcal{L}^2(\mu)} \to 0$, show that $\|H_n \bullet M_1 - K_n \bullet M_1\|_{\mathcal{L}^2(\mathbb{P})} \to 0$. Deduce that $K_n \bullet M_1$ also converges to Z in $\mathcal{L}^2(\mathbb{P})$ norm.

(iii) Define (up to an almost sure equivalence) $G \bullet M_1 = Z$.

(iv) For each t in $[0, 1]$, argue similarly that $G \bullet M_t$ could be defined as an $\mathcal{L}^2(\mathbb{P})$ limit of $H_n \bullet M_t$.

(v) Show that $\mathbb{P}_t(G \bullet M_1) = G_t$ almost surely. Choose a cadlag version of the martingale $\{(G \bullet M_t, \mathcal{F}_t) : 0 \le t \le 1\}$. Show that $G \mapsto G \bullet M$ is linear (up to some sort of almost sure equivalence).

(vi) Show that $\|G \bullet M\|_{\mathcal{M}} := \|G \bullet M_1\|_{\mathcal{L}^2(\mathbb{P})} = \|G\|_{\mathcal{L}^2(\mu)}$.

(vii) Suppose $\{G_i\}$ is a sequence in $\overline{\mathcal{H}}_{\text{simple}}$ with $\|G_i \bullet M_1 - W\|_{\mathcal{L}^2(\mathbb{P})} \to 0$ for some $W \in \mathcal{L}^2(\mathbb{P})$. Show that $\|G_i - G\|_{\mathcal{L}^2(\mu)} \to 0$ for some $G \in \overline{\mathcal{H}}_{\text{simple}}$ and $W = G \bullet M_1$ almost surely.

Deduce that $\{G \bullet M : G \in \overline{\mathcal{H}}_{\text{simple}}\}$ is a $\|\cdot\|_{\mathcal{M}}$ closed subspace of $\mathcal{M}_0^2$. [Strictly speaking, we should work with equivalence classes of indistinguishable processes.]

(viii) Show that the $H \bullet M$ described in Theorem 6 is unique up to indistinguishability.

Lect 12, Wednesday 17 Feb

<9> **Example.** Suppose σ and τ are stopping time taking values in $[0,1]$, with $\sigma \le \tau$. Suppose h is a bounded, $\mathcal{F}_\sigma$-measurable random variable. Define σ_n to be the stopping time obtained by rounding σ up to the next integer multiple of 2^{-n} and define τ_n similarly.

- Show that $\mu h(\omega)\left((\!(\sigma_n,\tau_n]\!] - (\!(\sigma,\tau]\!]\right)^2 \to 0$.
- Deduce that $\left(h(\omega)(\!(\sigma,\tau]\!]\right) \bullet M_t = h(\omega)\left(M_{t\wedge\tau} - M(\sigma\wedge t)\right)$.

□

6.6 Predictable integrands

How large is $\overline{\mathcal{H}}_{\text{simple}}$?

(i) Invoke a λ-space argument to show that $\overline{\mathcal{H}}_{\text{simple}}$ contains all bounded, $\mathcal{P}$-measurable processes.

(ii) In general, if $G \in \overline{\mathcal{H}}_{\text{simple}}$ then $\|G - H_n\|_{\mathcal{L}^2(\mu)} \to 0$ for some sequence $\{H_n\}$ in $\mathcal{H}_{\text{simple}}$. There exists a subsequence along which $H_n \to G$ a.e. $[\mu]$. As each H_n is $\mathcal{P}$-measurable, there must exists some $\mathcal{P}$-measurable G^* with $G^* = G$ a.e. $[\mu]$.

(iii) For Brownian motion, that $\overline{\mathcal{H}}_{\text{simple}}$ contains all the $\mathcal{F}_1\otimes\mathcal{B}(0,1]$-measurable, adapted processes that are square integrable for $\mathbb{P}\times\mathfrak{m}$. (See Chung and Williams 1990, Theorem 3.7 or Problem [2] below.) We don't lose much by restricting ourselves to $\mathcal{P}$-measurable integrands.

6.7 Problems

[1] Suppose $M \in \mathcal{M}^2[0,1]$ has continuous sample paths.

(i) For each H in $\mathcal{H}_{\text{simple}}$, show that $H \bullet M$ has continuous sample paths.

(ii) Suppose $\{H_n : n \in \mathbb{N}\} \subseteq \mathcal{H}_{\text{simple}}$ and $\mu|H_n - H|^2 < 2^{-n}$. Use Doob's maximal inequality to show that

$$\sum\nolimits_n \mathbb{P}\sup\nolimits_{0\le t\le 1}|H_n \bullet M_t - H \bullet M_t| < \infty$$

(iii) Deduce that there is a version of $H \bullet M$ with continuous sample paths.

[2] Suppose $\mu = \mathfrak{m} \otimes \mathbb{P}$, defined on $\mathcal{B}(0,1] \otimes \mathcal{F}$. Let $\{X_t : 0 \le t \le 1\}$ be progressively measurable.

(i) Suppose X is bounded, that is, $\sup_{t,\omega} |X(t,\omega)| < \infty$. Define

$$H_n(t,\omega) := n \int_{t-n^{-1}}^{t} X(s,\omega)\, ds$$

(How should you understand the definition when $t < n^{-1}$?) Show that H_n is predictable and $\int_0^1 |H_n(t,\omega) - X(t,\omega)|^2\, dt \to 0$ for each ω.

(ii) Deduce that $\mu |H_n - X|^2 \to 0$.

(iii) Deduce that $X \in \overline{\mathcal{H}}_{\text{simple}}$ if $\mu X^2 < \infty$.

References

Chung, K. L. and R. J. Williams (1990). *Introduction to Stochastic Integration*. Boston: Birkhäuser.

Dellacherie, C. and P. A. Meyer (1978). *Probabilities and Potential*. Amsterdam: North-Holland. (First of three volumes).

Dellacherie, C. and P. A. Meyer (1982). *Probabilities and Potential B: Theory of Martingales*. Amsterdam: North-Holland.

Métivier, M. (1982). *Semimartingales: A Course on Stochastic Processes*. Berlin: De Gruyter.

Pollard, D. (2001). *A User's Guide to Measure Theoretic Probability*. Cambridge University Press.

Rogers, L. C. G. and D. Williams (1987). *Diffusions, Markov Processes, and Martingales: Itô Calculus*, Volume 2. Wiley.