Project 7

Localization of the stochastic integral

In this project you will learn how to extend $H \bullet M$ to a larger class of processes indexed by $\mathbb{R}^+$, by working with stopped processes. Most processes will be defined on a fixed (complete) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$.

Lect 13, Monday 22 February

7.1 Notation and definitions

My intention is to establish a correspondence between processes indexed by $[0,\infty]$ and processes indexed by $[0,1]$, so that the theory of the isometric stochastic integral extends without much effort to a set of martingales indexed by $\mathbb{R}^+$.

<1> **Definition.** *Write $\mathcal{M}^2(\mathbb{R}^+)$ for the set of all **square-integrable martingales** indexed by $\mathbb{R}^+$, that is, the cadlag martingales $\{M_t : t \in \mathbb{R}^+\}$ for which $\sup_t \mathbb{P}M_t^2 < \infty$. Define $\mathcal{M}_0^2(\mathbb{R}^+) = \{M \in \mathcal{M}_0^2(\mathbb{R}^+) : M_0 \equiv 0\}$.*

However, I am not completely confident that I have discovered all the subtleties involved in behavior of processes near 0 and ∞. To keep my options open I will try to distinguish carefully between four possible sets on which processes might be defined: needed?

$$
\begin{aligned}
\mathfrak{S}_\infty^\circ :=&\quad \Omega \times (0,\infty) = \{(\omega,t) : \omega \in \Omega,\ 0 < t < \infty\} \\
&\quad \Omega \times \mathbb{R}^+ = \{(\omega,t) : \omega \in \Omega,\ 0 \le t < \infty\} \\
\mathfrak{S}_\infty :=&\quad \Omega \times (0,\infty] = \{(\omega,t) : \omega \in \Omega,\ 0 < t \le \infty\} \\
\overline{\mathfrak{S}}_\infty :=&\quad \Omega \times \overline{\mathbb{R}}^+ = \{(\omega,t) : \omega \in \Omega,\ 0 \le t \le \infty\}
\end{aligned}
$$

As before, define $\mathcal{F}_\infty := \sigma\left(\cup_{t\in\mathbb{R}^+} \mathcal{F}_t\right)$, although I wonder whether $\mathcal{F}_{\infty-}$ might be a better notation.

<2> **Definition.** *Write $\mathcal{T}$ for the set of all stopping times, with values in $[0,\infty]$, for the filtration.*

*(i) If X is a process on $\Omega\times\mathbb{R}^+$ and $\tau \in \mathcal{T}$, define the **stopped process** $X_{\wedge\tau}$ (nonstandard notation) by*

$$X_{\wedge\tau}(t,\omega) := X(\tau(\omega)\wedge t, \omega) \qquad \text{for } t \in \mathbb{R}^+.$$

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(ii) *Call a process X on $\mathfrak{S}_\infty$ and $\mathbb{L}$-process if it is adapted and all its sample paths are left-continuous at each t in $(0,\infty]$ with finite right limits at each t in $[0,\infty)$.*

> **Remark.** *Rogers and Williams (1987, page 1) defined an L-process indexed by $(0,\infty)$ to have paths that are "left-continuous with limits from the right", which I interpret to mean that no special behavior is assumed near 0 or ∞. I suspect that my requirement of left-continuity at ∞ is not essential, although it does ensure that $X(\cdot,\infty)$ is $\mathcal{F}_\infty$-measurable. Existence of a finite right limit at 0 ensures that my $\mathbb{L}$-processes are locally bounded. See Section 5.*

(iii) *The predictable sigma-field $\mathcal{P}_\infty$ on $\mathfrak{S}_\infty$ is defined as the sigma-field generated by all $\mathbb{L}$-processes on $\mathfrak{S}_\infty$.*

(iv) *For each pair of stopping times $\sigma \le \tau$ taking values in $[0,\infty]$ define the stochastic interval $((\sigma,\tau]] := \{(\omega,t) \in \Omega \times \mathbb{R}^+ : \sigma(\omega) < t \le \tau(\omega)\}$.*

> **Remark.** *Notice that the definition excludes (ω,∞) from the stochastic interval even when $\tau(\omega) = \infty$. In particular, for $\sigma \equiv 0$ and $\tau \equiv \infty$ we get $((0,\infty]] = \Omega \times \mathbb{R}^+$. Don't be misled by the "$\infty]]$" into assuming that $\Omega \times \{\infty\}$ is included. The convention that ∞ is excluded makes possible some neat arguments, even though it spoils the analogy with stochastic subintervals of $(0,1] \times \Omega$. Although sorely tempted to buck tradition, I decided to stick with established usage for fear of unwanted exceptions to established theorems.*

After writing all these definitions I feel like a lawyer who is worried about the interpretation of every single comma in a legal document. Please don't sue me if my latest attempt at precision still doesn't work.

7.2 Stopped processes indexed by $[0,1]$

Suppose $M \in \mathcal{M}_0^2[0,1]$ and $H \in \mathcal{H}^2(\mu_{M^2})$, where μ_{M^2} is the Doléans measure on the predictable sigma-field on $\mathfrak{S}_1 := \Omega \times (0,1]$ defined by the submartingale M_t^2. Let τ be a $[0,1]$-valued stopping time. Let X denote the martingale $H \bullet M$.

(i) Define $N = M_{\wedge\tau}$. Use Doob's inequality to show that $N \in \mathcal{M}_0^2[0,1]$.

(ii) Suppose $H \in \mathcal{H}_{\text{simple}}$. Show that, with probability one,

$$X_{t\wedge\tau} = \big(H((0,\tau]] \bullet M\big)_t = H \bullet N_t \qquad \text{for } 0 \le t \le 1.$$

You can use Example 9 from Project 6 to handle $H((0,\tau]]$.

(iii) Extend the previous part to general H in $\mathcal{H}^2(\mu_M)$, by considering sequences $\{H_n\}$ in $\mathcal{H}_{\text{simple}}$ for which $\mu_M|H_n - H|^2 \to 0$.

(iv) Show that the Doléans measure μ_X for X has density H^2 with respect to μ_M. Remember that μ_X is uniquely determined by $\mu_X((0,\tau]] = \mathbb{P}X^2_{\tau\wedge 1}$, as τ run over all stopping times.

(v) Suppose $K \in \mathcal{H}^2(\mu_X)$. Show that $KH \in \mathcal{H}^2(\mu_M)$ and $K \bullet (H \bullet M) = (HK) \bullet M$. Hint: Use a λ-space argument, starting from stochastic intervals $\{((0,\tau]]$, to establish both assertions for bounded K's. Then approximate by $K_n \in \mathcal{H}_{\text{simple}}$ with $2|K| \geq |K_n - K| \to 0$ pointwise.

7.3 Square-integrable martingales: indexed by $\mathbb{R}^+$ or $[0,\infty]$?

Suppose $M \in \mathcal{M}^2(\mathbb{R}^+)$. Notice that M_∞ is not yet defined.

(i) For each $i \in \mathbb{N}$ define $\xi_i := M_i - M_{i-1}$ and $v_i := \mathbb{P}\xi_i^2$. Also define $V_i := v_1 + \cdots + v_i$ and $V_\infty := \sup_{i\in\mathbb{N}} V_i$, which is finite.

(ii) For each $n, m \in \mathbb{N}$ define $\Delta_{n,m} := \sup\{|M_s - M_t| : n \leq s, t \leq m\}$ and $\Delta_{n,\infty} := \sup_{m\in\mathbb{N}} \Delta_{n,m}$. Use Doob's inequality to show that

$$\mathbb{P}\Delta^2_{n,\infty} \leq 16(V_\infty - V_n).$$

(iii) Deduce that $\Delta_{n,\infty} \downarrow \Delta_\infty = 0$ almost surely.

(iv) Show that there exists an $M_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ such that $M_t \to M_\infty$ almost surely and $\mathbb{P}|M_t - M_\infty|^2 \to 0$ as $t \to \infty$.

(v) Show that $M_t = \mathbb{P}(M_\infty \mid \mathcal{F}_t)$ almost surely.

(vi) Conclude that $\{(M_t, \mathcal{F}_t) : 0 \leq t \leq \infty\}$ is a cadlag martingale with $\sup_{0\leq t\leq\infty} \mathbb{P}M_t^2 = \mathbb{P}M_\infty^2 < \infty$ with sample paths that are left continuous at ∞.

The one-to-one map $s = \psi(t) := t/(1+t)$ with $\psi(\infty) = 1$ lets us identify processes indexed by $[0,\infty]$ with processes indexed by $[0,1]$. The filtration $\{\mathcal{F}_t : 0 \leq t \leq \infty\}$ is carried into a filtration $\{\mathcal{G}_s : 0 \leq s \leq 1\}$ with $\mathcal{G}_{\psi(t)} := \mathcal{F}_t$.

The correspondence $N_{\psi(t)} = M_t$ identifies a martingale M in $\mathcal{M}^2(\mathbb{R}^+)$ with a martingale $\{N_s : 0 \le s \le 1\}$ in $\mathcal{M}^2[0,1]$.

Most of the theory for the isometric stochastic integrals with respect to $\mathcal{M}_0^2[0,1]$ processes carries over to analogous theory for $\mathcal{M}^2(\mathbb{R}^+)$, with a few subtle differences. For $\mathcal{M}^2(\mathbb{R}^+)$ we have left continuity of sample paths at ∞, by construction of M_∞; for $\mathcal{M}^2[0,1]$ we did not require left continuity at 1. Also we did not require that $\mathcal{F}_1 = \sigma\left(\cup_{t<1}\mathcal{F}_t\right)$.

> **Remark.** A better analogy might allow the sigma-field $\mathcal{F}_\infty$ to be larger than $\mathcal{F}_{\infty-} := \sigma\left(\cup_{t\in\mathbb{R}^+}\mathcal{F}_t\right)$ and might allow M to have a jump at ∞.

(i) The stochastic interval $((0,\tau]]$ on $\mathfrak{S}_\infty$ contains no points of $\Omega \times \{\infty\}$, even if τ might take infinite values; the corresponding stochastic interval $((0,\psi(\tau)]]$ is allowed to contains points in $\Omega \times \{1\}$.

(ii) Luckily, the Doléans measure μ_{N^2}, which is defined on the predictable sigma-field of $\mathfrak{S} := \Omega \times (0,1]$, puts zero mass on $\Omega \times \{1\}$ because

$$\begin{aligned} \mu_{N^2}((0,\psi(n)]] &= \mathbb{P}N^2_{\psi(n)} \\ &= \mathbb{P}M_n^2 \\ &\to \mathbb{P}M_\infty^2 \qquad \text{as } n\to\infty \\ &= \mathbb{P}N_1^2 \\ &= \mu_{N^2}((0,1]]. \end{aligned}$$

Thus the map $(\omega,s) \mapsto (\omega,\psi^{-1}(s))$ from $\mathfrak{S}$ onto $\mathfrak{S}_\infty$ carries μ_{N^2} onto a measure μ_{M^2} that concentrates on the set $\Omega \times (0,\infty)$, and we still have

$$\mathbb{P}N^2_{\psi(\tau)} = \mu_{N^2}((0,\psi(\tau)]] = \mu_{M^2}((0,\tau]] = \mathbb{P}M_\tau^2$$

for every stopping time τ.

Lect 14, Wednesday 24 February

7.4 Locally square-integrable martingales

<3> **Definition.** *A process $\{M_t : t \in \mathbb{R}^+\}$ is said to be a* ***locally square-integrable martingale*** *if there exists a sequence of stopping times $\{\tau_k\}$ with $\tau_k(\omega) \uparrow \infty$ for each ω and $M_{\wedge\tau_k} \in \mathcal{M}_0^2(\mathbb{R}^+)$ for each k. Write* $\mathrm{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ *for the set of all such processes. The stopping times are called a* ***localizing sequence*** *for M.*

Remark. My definition of a locally square-integrable martingale agrees with that of Dellacherie and Meyer (1982, page 228), but differs slightly from that of Métivier (1982, page 148), who does not require $M_0 \equiv 0$.

Notice that if $\{\tau_k\}$ is a localizing sequence for M then so is $\{k \wedge \tau_k\}$. Just to be on the safe side, I will always assume each τ_k is a bounded stopping time.

The Doléans measure for an M in $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$.

Let $\{\tau_k : k \in \mathbb{N}\}$ be a localizing sequence for M. Let μ_k be the Doléans measure for $M_{\wedge\tau_k}$, so that $\mu_k((0,\tau]] = \mathbb{P}M^2_{\tau\wedge\tau_k}$ for each stopping time τ.

(i) Show that $\mu_k((0,\infty]] = \mathbb{P}M^2_{\tau_k} = \mu_k((0,\tau_k]]$. Deduce that μ_k puts zero mass on $((\tau_k,\infty]]$.

(ii) For each stopping time τ, show that

$$\mu_{k+1}((0,\tau\wedge\tau_k]] = \mathbb{P}M^2_{\tau\wedge\tau_k} = \mu_k((0,\tau]].$$

Deduce that μ_k equals the restriction of μ_{k+1} to $((0,\tau_k]]$.

(iii) Show that there exists a sigma-finite measure μ on $\mathcal{P}_\infty$ whose restriction to $((0,\tau_k]]$ equals μ_k, for each k.

(iv) For each (bounded?) stopping time σ for which $M_{\wedge\sigma} \in \mathcal{M}_0^2(\mathbb{R}^+)$ show that $\mu((0,\sigma]] = \mathbb{P}M^2_\sigma$. Conclude that μ does not depend on the choice of localizing sequence for M.

7.5 Locally bounded predictable processes

Write $\mathcal{H}_{\text{Bdd}}$ for the set of all $\mathcal{P}_\infty$-measurable processes H on $\mathfrak{S}_\infty$ that are bounded in absolute value by some finite constant (depending on H).

<4> **Definition.** *Define* $\text{loc}\mathcal{H}_{\text{Bdd}}$ *to be the set of all predictable processes H for which there exists a localizing sequence of stopping times $\{\tau_k : k \in \mathbb{N}\}$ for which $H_{\wedge\tau_k} \in \mathcal{H}_{\text{Bdd}}$ for each k.*

Say that a sequence $\{H^{(i)} : i \in \mathbb{N}\}$ of $\text{loc}\mathcal{H}_{\text{Bdd}}$ processes is ***locally uniformly bounded*** *if there exists a single localizing sequence of stopping times $\{\tau_k : k \in \mathbb{N}\}$ and a sequence of finite constants $\{C_k\}$ such that $|H^{(i)}(\omega, t\wedge\tau_k(\omega))| \le C_k$ for all i and all $(\omega,t) \in \mathfrak{S}_\infty$.*

(i) Show that every $\mathbb{L}$-process X belongs to $\text{loc}\mathcal{H}_{\text{Bdd}}$. Hint: Consider $\tau_k(\omega) := \inf\{t \in \mathbb{R}^+ : |X_t(\omega)| \ge k\}$.

(ii) (Much harder) Is the previous assertion still true if we replace $\mathbb{L}$-processes by $\mathcal{P}$-measurable processes? What if we also require each sample path to be cadlag?

> **Remark.** A complete resolution of this question requires some facts about predictable stopping times and predictable cross-sections. Compare with Métivier (1982, Section 6) or Dellacherie and Meyer (1982, VII.32).

Lect 15, Monday 1 March

7.6 Localization of the isometric stochastic integral

The new stochastic integral will be defined indirectly by a sequence of isometries. The continuity properties of $H \bullet M$ will be expressed not via $\mathcal{L}^2$ bounds but by means of the concept of ***uniform convergence in probability on compact intervals***.

<5> **Definition.** *For a sequence of processes* $\{Z_n\}$, *write* $Z_n \overset{ucpc}{\longrightarrow} Z$ *to mean that*

$$\sup_{0 \le s \le t} |Z_n(\omega, s) - Z(\omega, s)| \to 0 \qquad \textit{in probability,}$$

for each t *in* $\mathbb{R}^+$.

<6> **Theorem.** *Suppose* $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. *There exists a linear map* $H \mapsto H \bullet M$ *from* $\text{loc}\mathcal{H}_{\text{Bdd}}$ *into* $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ *with the following properties.*

(a) $((0, \tau]] \bullet M_t = M_{t \wedge \tau}$ *for all* $\tau \in \mathcal{T}$.

(b) $(H \bullet M)_{t \wedge \tau} = \big(H((0, \tau]]\big) \bullet M_t = \big(H \bullet M_{\wedge \tau}\big)_t$, *for all* $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ *and all* $\tau \in \mathcal{T}$.

(c) If M *has continuous sample paths then so does* $H \bullet M$.

(d) Suppose $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ *is locally uniformly bounded and* $H^{(n)}(t, \omega) \to 0$ *for each* (t, ω). *Then* $H^{(n)} \bullet M \overset{ucpc}{\longrightarrow} 0$.

Proof (Sketch) Suppose M has localizing sequence $\{\tau_k : k \in \mathbb{N}\}$ and $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ has localizing sequence $\{\sigma_k : k \in \mathbb{N}\}$.

(i) Why is there no loss of generality in assuming that $\sigma_k = \tau_k$ for every k?

(ii) Write $M^{(k)}$ for $M_{\wedge \tau_k}$. Define $X^{(k)} = \big(H((0, \tau_k]]\big) \bullet M^{(k)}$.

(iii) Show that $X^{(k)}(t,\omega) = X^{(k)}(t \wedge \tau_k(\omega), \omega)$ for all $t \in \mathbb{R}^+$. That is, show that the sample paths are constant for $t \geq \tau_k(\omega)$. Do we need some sort of almost sure qualification here?

(iv) Show that, on a set of ω with probability one,

$$X^{(k+1)}(t \wedge \tau_k(\omega), \omega) = X^{(k)}(t \wedge \tau_k(\omega), \omega) \qquad \text{for all } t \in \mathbb{R}^+.$$

(v) Show that there is a cadlag adapted process X for which, on a set of ω with probability one,

$$X(t \wedge \tau_k(\omega), \omega) = X^{(k)}(t \wedge \tau_k(\omega), \omega) \qquad \text{for all } t \in \mathbb{R}^+, \text{ all } k.$$

(vi) Show that $X \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$, with localizing sequence $\{\tau_k : k \in \mathbb{N}\}$.

(vii) Define $H \bullet M := X$.

(viii) In order to establish linearity of $H \mapsto H \bullet M$, you need to show that the definition does not depend on the particular choice of the localizing sequence. (If we can use a single localizing sequence for two different H processes then linearity for the approximating $X^{(k)}$ processes will transfer to the X process.)

(ix) For assertion (d), we may also assume that $\{\tau_k\}$ localizes M to $\mathcal{M}_0^2(\mathbb{R}^+)$. Write μ_k for the Doléans measure of the submartingale $(M^{(k)})^2$. Then, for each fixed k, we have

$$\begin{aligned}
&\mathbb{P}\sup\nolimits_{s\leq t}\left(H^{(n)} \bullet M\right)^2_{s\wedge\tau_k} \\
&\qquad = \mathbb{P}\sup\nolimits_{s\leq t}\left(H^{(n)}((0,\tau_k]] \bullet M_s^{(k)}\right)^2 \qquad \text{by construction} \\
&\qquad \leq 4\mathbb{P}\left(H^{(n)}((0,\tau_k]] \bullet M_t^{(k)}\right)^2 \qquad \text{by Doob's inequality} \\
&\qquad = 4\mu_k\left((H^{(n)})^2((0,\tau_k \wedge t]]\right) \\
&\qquad \to 0 \qquad \text{as } n \to \infty, \text{ by Dominated Convergenece.}
\end{aligned}$$

When $\tau_k > t$, which happens with probability tending to one, the processes $H^{(n)} \bullet M_{s\wedge\tau_k}$ and $H^{(n)} \bullet M_s$ coincide for all $s \leq t$. The uniform convergence in probability follows.

□

7.7 Characterization of the stochastic integral

I haven't checked carefully whether the following result is still true after all the notational changes I have been making.

<7> **Theorem.** *Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and $\mathcal{I} : \text{loc}\mathcal{H}_{\text{Bdd}} \to \mathcal{M}_0^2(\mathbb{R}^+)$ is a linear map (in the sense of indistinguishability of processes) for which*

(i) $\mathcal{I}\,(\,((0,\tau]]\,)_t = M_{\tau\wedge t}$ almost surely, for each $t \in \mathbb{R}^+$ and $\tau \in \mathcal{T}$.

(ii) For each locally uniformly bounded sequence $\{H^{(n)} : n \in \mathbb{N}\}$ in $\text{loc}\mathcal{H}_{\text{Bdd}}$ that converges to zero pointwise, $\mathcal{I}(H^{(n)})_t \to 0$ in probability for each t.

Then $\mathcal{I}(H)_t = H \bullet M_t$ almost surely for each $t \in \mathbb{R}^+$ and each $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

Remark. The assertion of the Theorem can also be written: there exists a set Ω_0 with $\Omega_0^c \in \mathcal{N}$ such that

$$\psi(H)(t,\omega) = H \bullet M(t,\omega) \qquad \text{for every } t \text{ if } \omega \in \Omega_0$$

Cadlag sample paths allow us to deduce equality of whole paths (indistinguishability) from equality at a countable dense set of times.

Proof Use a λ-space argument to prove that $\mathcal{I}(H)_t = H \bullet M_t$ almost surely, for each H in $\mathcal{H}_{\text{Bdd}}$.

For an H in $\text{loc}\mathcal{H}_{\text{Bdd}}$ with localizing sequence $\{\tau_k\}$, show that the sequence of processes $H^{(k)} := H((\tau_k, \infty]]$ is locally uniformly bounded and it converges pointwise to zero. Deduce that

$$\mathcal{I}(H)_t - H((0,\tau_k]] \bullet M_{t\wedge\tau_k} = \mathcal{I}(H^{(k)})_t \to 0 \qquad \text{in probability.}$$

Then show that

$$H((0,\tau_k]] \bullet M_{t\wedge\tau_k} \to H \bullet M_t \qquad \text{in probability, as } k \to \infty.$$

<8> **Example.** Show that $K \bullet (H \bullet M) = (KH) \bullet M$ for all $K, H \in \text{loc}\mathcal{H}_{\text{Bdd}}$, by the following argument.

(i) Define $X = H \bullet M$ and define $\mathcal{I}$ by $\mathcal{I}(H) := (KH) \bullet M$.

(ii) Show that, with probability one,

$$\mathcal{I}\,(\,((0,\tau]]\,)_t = (H \bullet M)_{t\wedge\tau} = X_{t\wedge\tau} = H((0,\tau]] \bullet X_t$$

(iii) Use Theorem 7.

References

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