# Project 8

## Semimartingales

For this project I found Protter (1990, Chapter II) very useful.

Once again, all random variables will be defined on a fixed (complete) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$. After much experimentation in Project 7, I have decided all the fussing about $+\infty$ is too much of a distraction. For this project, an $R$-process will be an adapted process with cadlag sample paths, defined on

$$\mathfrak{S} = \Omega \times \mathbb{R}^+ := \{(\omega, t) : \omega \in \Omega, \, 0 \le t < \infty\}.$$

The predictable sigma-field $\mathcal{P}$ and predictable processes will be defined on $\mathfrak{S}^\circ := \Omega \times (0, \infty)$. An $L$-process will be an adapted process, defined on $\mathfrak{S}^\circ$, for which each sample path is left-continuous at each point of $(0, \infty)$. Here is the (slightly) revised form of the definitions from Project 7.

<1> **Definition.**

*(i) Write $\mathcal{T}$ for the set of all stopping times with values in $[0, \infty]$*

*(ii) Write $\mathcal{M}^2(\mathbb{R}^+)$ for the set of all **square-integrable martingales** indexed by $\mathbb{R}^+$, that is, the cadlag martingales $\{M_t : t \in \mathbb{R}^+\}$ for which $\sup_t \mathbb{P}M_t^2 < \infty$. Define $\mathcal{M}_0^2(\mathbb{R}^+) = \{M \in \mathcal{M}_0^2(\mathbb{R}^+) : M_0 \equiv 0\}$.*

*(iii) A **localizing sequence** is a set of stopping times $\{\tau_k : k \in \mathbb{N}\}$ for which $\tau_k(\omega) \to \infty$ as $k \to \infty$.*

*(iv) Write $\mathrm{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ for the set of all **locally square-integrable martingales**: $R$-processes $M$ with $M_0 \equiv 0$ and $M_{\wedge\tau_k} \in \mathcal{M}_0^2(\mathbb{R}^+)$ for each $k$, for some localizing sequence $\{\tau_k\}$.*

*(v) Write $\mathcal{H}_{\mathrm{Bdd}}$ for the set of all predictable processes $H$ on $\mathfrak{S}^\circ$ that are bounded in absolute value by some finite constant (depending on $H$).*

*(vi) Define $\mathrm{loc}\mathcal{H}_{\mathrm{Bdd}}$ to be the set of all predictable processes $H$ for which there exists a localizing sequence of stopping times $\{\tau_k : k \in \mathbb{N}\}$ for which $H((0, \tau_k]] \in \mathcal{H}_{\mathrm{Bdd}}$ for each $k$. That is, for each $k$ there exists a finite constant $C_k$ such that $|H(\omega, t)\{0 < t \le \tau_k(\omega)\}| \le C_k$ for all $(\omega, t)$ in $\mathfrak{S}^\circ$.*

*(vii) Say that a sequence* $\{H^{(i)} : i \in \mathbb{N}\}$ *of* $\text{loc}\mathcal{H}_{\text{Bdd}}$ *processes is* ***locally uniformly bounded*** *if there exists a single localizing sequence of stopping times* $\{\tau_k : k \in \mathbb{N}\}$ *and a sequence of finite constants* $\{C_k\}$ *such that* $|H^{(i)}(\omega, t)((0, \tau_k]]| \le C_k$ *for all* $i$ *and all* $(\omega, t) \in \mathfrak{S}^\circ$.

Lect 16, Wednesday 3 March

## 8.1 Processes of finite variation as random (signed) measures

A real-valued function $f : \mathbb{R}^+ \to \mathbb{R}$ is said to be of finite variation on $\mathbb{R}^+$ if for each $t$ in $\mathbb{R}^+$ there exists a finite constant $C_t$ such that

$$\sum\nolimits_{i=1}^{N} |f(t_i) - f(t_{i-1})| \le C_t$$

for all finite partitions $0 = t_0 < t_1 < \cdots < t_N = t$ of $[0, t]$. As shown in Appendix D, a function $f$ is of finite variation if and only if it can written as a difference of two nondecreasing functions, $f(t) = g_1(t) - g_2(t)$. Moreover, if $f$ is cadlag, then both $g_i$ can be chosen as cadlag functions. In that case, each $g_i$ can also be thought of as a "distribution function" of a sigma-finite measure $\Gamma_i$ on $\mathcal{B}(\mathbb{R}^+)$:

$$\Gamma_i(0, t] = g_i(t) - g_i(0) \qquad \text{for each } t \in \mathbb{R}^+.$$

We could think of $g_i(0)$ as the mass placed by $\Gamma_i$ at 0. If we assume $f(0) = g_1(0) = g_2(0)$ then

$$f(t) = \Gamma_1[0, t] - \Gamma_2[0, t] \qquad \text{for each } t \in \mathbb{R}^+.$$

The difference $\Gamma_1 - \Gamma_2$ is a countably additive signed measure.

*<2>* **Definition.** *Write* $\mathbb{FV}_0 = \mathbb{FV}_0(\mathbb{R}^+)$ *for the set of all R-processes* $A$ *on* $\mathfrak{S}$ *for which* $A(\omega, 0) \equiv 0$ *and for which each sample path* $A(\omega, \cdot)$ *is of finite variation on* $\mathbb{R}^+$. *Equivalently,* $\mathbb{FV}_0$ *consists of all processes expressible as a difference* $A = L_1 - L_2$ *of two R-processes for which* $L_1(\omega, 0) = L_2(\omega, 0) \equiv 0$ *and for which each* $L_i(\omega, \cdot)$ *is an increasingfunction on* $\mathbb{R}^+$.

The stochastic integral with respect to $A$ will be defined as a difference of stochastic integrals with respect to $L_1$ and $L_2$. Questions of uniqueness—lack of dependence on the choice of the two increasing processes—will be subsumed in the the uniqueness assertion for semimartingales.

The case where $\{L_t : t \in \mathbb{R}^+\}$ is an R-process with nondecreasing sample paths and $L_0 \equiv 0$ will bring out the main ideas. I will leave to you the task of extending the results to a difference of two such processes. Each sample path of $L$ defines a sigma-finite measure $\lambda_\omega$ on $\mathcal{B}(\mathbb{R}^+)$,

$$\lambda_\omega[0,t] = L(\omega,t) \qquad \text{for } t \in \mathbb{R}^+.$$

Notice that $\lambda_\omega\{0\} = L(\omega,0) = 0$ The family $\Lambda = \{\lambda_\omega : \omega \in \Omega\}$ may be thought of as a ***random measure***, that is, a map from $\Omega$ into the space of (sigma-finite) measures on $\mathcal{B}(\mathbb{R}^+)$.

Define the stochastic integral of a process $H$ on $\mathfrak{S}$ with respect to $L$ pathwise,

$$H \bullet L_t := \lambda_\omega^s \left(\{0 < s \le t\} H(\omega,s)\right).$$

This integral is well defined at least if $H(\omega,\cdot)$ is measurable and bounded on each interval $[0,t]$.

**Remark.** Should I be more careful about where $H$ is defined? Does my precaution of making $\lambda_\omega\{0\} = 0$ take care of any ambiguities if $H$ is only defined on $\mathfrak{S}^\circ$?

In particular, $H \bullet L$ is well defined if $H \in \mathrm{loc}\mathcal{H}_{\mathrm{Bdd}}$. In fact $H \bullet L \in \mathbb{FV}_0$. Indeed, for some localizing sequence $\{\tau_k\}$ there are finite constants $C_k$ for which $|H(\omega,s)\{0 < s \le \tau_k(\omega)\}| \le C_k$ for all $(\omega,s)$ in $\mathfrak{S}^\circ$. For each fixed $\omega$, the function $s \mapsto H(\omega,s)$ is measurable (by Fubini, because predictable implies progressively measurable) and is, therefore, integrable with respect to $\lambda_\omega$ on each bounded interval. By Dominated Convergence, the sample paths are cadlag. Also $H \bullet L_t = H^+ \bullet L_t - H^- \bullet L_t$, a difference of two nondecreasing R-processes.

You should now be able to prove the following result by using standard facts about measures.

<3> **Theorem.** *Suppose $A \in \mathbb{FV}_0$. There is a map $H \mapsto H \bullet A$ from $\mathrm{loc}\mathcal{H}_{\mathrm{Bdd}}$ to $\mathbb{FV}_0$ that is linear (in the almost sure sense?) for which:*

*(i)* $((0,\tau]] \bullet A_t = A_{t\wedge\tau}$ *for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$*

*(ii)* $(H \bullet A)_{t\wedge\tau} = (H((0,\tau]]) \bullet A_t = H \bullet (A_{\wedge\tau})_t$ *for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$*

*(iii) If $\{H_n : n \in \mathbb{N}\} \subset \mathrm{loc}\mathcal{H}_{\mathrm{Bdd}}$ is locally uniformly bounded and converges pointwise (in $\omega$ and $t$) to $H$ then $H \in \mathrm{loc}\mathcal{H}_{\mathrm{Bdd}}$ and $H_n \bullet A \xrightarrow{ucpc} H \bullet A$.*

As you can see, there is really not much subtlety beyond the usual measure theory in the construction of stochastic integrals with respect to $\mathbb{FV}_0$-processes.

> **Remark.** The integral $H \bullet L_t$ can be defined even for processes that are not predictable or locally bounded. In fact, as there are no martingales involved in the construction, predictability is irrelevant. However, functions in $\text{loc}\mathcal{H}_{\text{Bdd}}$ will have stochastic integrals defined for both $\mathbb{FV}_0$-processes and $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$-processes.

## 8.2 Stochastic integrals with respect to semimartingales

By combining the results from the previous Section with results from Project 7, we arrive at a most satisfactory definition of the stochastic integral for a very broad class of processes.

<4> **Definition.** *An R-process $X$ is called a **semimartingale**, for a given standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$, if it can be decomposed as $X_t = X_0 + M_t + A_t$ with $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and $A \in \mathbb{FV}_0$. Write $\mathbb{SMG}$ for the class of all semimartingales and $\mathbb{SMG}_0$ for those semimartinagles with $X_0 \equiv 0$.*

$\mathbb{SMG}$ is nonstandard notation

Notice that $\mathbb{SMG}_0$ is stable under stopping. Moreover, every local semimartingale is a semimartingale, a fact that is surprisingly difficult (Dellacherie and Meyer 1982, §VII.26) to establish directly.

The stochastic integral $H \bullet X$ is defined (up to indistinguishability) as the sum of the stochastic integrals with respect to the components $M$ and $A$. The value $X_0$ plays no role in this definition, so we may as well assume $X \in \mathbb{SMG}_0$. The resulting integral inherits the properties shared by integrals with respect to $\mathbb{FV}_0$ and integrals with respect to $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$.

> **Remark.** The stochastic integral will only be defined up to indistinguishability. Remember that two processes $Y$ and $Z$ are said to be indistinguishable if there exists a single $\mathbb{P}$-negligible set $N$ such that $Y(\omega, t) = Z(\omega, t)$ for all $(\omega, t)$ in $N^c \times \mathbb{R}^+$. I will also said that $Y$ and $Z$ are equal almost pathwise.

<5> **Theorem.** *For each $X$ in $\mathbb{SMG}_0$, there is a linear (modulo indistinguishability) map $H \mapsto H \bullet X$ from $\text{loc}\mathcal{H}_{\text{Bdd}}$ into $\mathbb{SMG}_0$ such that:*

*(i)* $((0, \tau]] \bullet X_t = X_{t \wedge \tau}$ *for each* $\tau \in \mathcal{T}$ *and* $t \in \mathbb{R}^+$.

*(ii)* $H \bullet X_{t \wedge \tau} = (H((0, \tau]]) \bullet X_t = H \bullet (X_{\wedge \tau})_t$ *for each* $\tau \in \mathcal{T}$ *and* $t \in \mathbb{R}^+$.

*(iii) If* $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ *is locally uniformly bounded and converges pointwise to $H$, then* $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ *and* $H^{(n)} \bullet X \xrightarrow{ucpc} H \bullet X$.

*Conversely, let $\mathcal{I}$ be another linear map from* $\text{loc}\mathcal{H}_{\text{Bdd}}$ *into the set of R-processes having at least the weaker properties:*

*(iv)* $\mathcal{I}(\,((0,\tau]]\,)_t = X_{t\wedge\tau}$ *almost surely, for each* $\tau\in\mathcal{T}$ *and* $t\in\mathbb{R}^+$.

*(v) If* $\{H^{(n)} : n\in\mathbb{N}\}\subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ *is locally uniformly bounded and converges pointwise to* $0$ *then* $\mathcal{I}(H^{(n)})_t\to 0$ *in probability, for each fixed* $t$.

*Then* $\mathcal{I}(H)_t = H\bullet X_t$ *almost surely for every* $t$.

**Remarks.** The converse shows, in particular, that the stochastic integral $H\bullet X$ does not depend on the choice of the processes $M$ and $A$ in the semimartingale decomposition of $X$.

PROOF (Outline for the converse) Define

$$\mathcal{H} := \{H\in\mathcal{H}_{\text{Bdd}} : \mathcal{I}(H)_t = H\bullet X_t \text{ almost surely, for each } t\in\mathbb{R}^+\}$$

(a) Show that $((0,\tau]]\in\mathcal{H}$, for each $\tau\in\mathcal{T}$.

(b) Show that $\mathcal{H}$ is a $\lambda$-space. Hint: If $H^{(n)}\in\mathcal{H}$ and $H^{(n)}\uparrow H$, with $H$ bounded, apply (iii) and (v) to $\{H-H^{(n)}\}$, which is uniformly bounded.

(c) Deduce that $\mathcal{H}$ equals $\mathcal{H}_{\text{Bdd}}$.

(d) Extend the conclusion to $\text{loc}\mathcal{H}_{\text{Bdd}}$. Hint: If $H\in\text{loc}\mathcal{H}_{\text{Bdd}}$, with $|H((0,\tau_k]]|\le C_k$, show that the processes $H^{(n)} := H((0,\tau_n]]$ are locally uniformly bounded and converge pointwise to $H$.

□

I have found the properties of the stochastic integral asserted by the Theorem to be adequate for many arguments. I consider it a mark of defeat if I have to argue separately for the $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and $\mathbb{FV}_0$ cases to establish a general result about semimartingales.

**Remark.** If we know that the $H$ in Theorem 5(iii) is predictable then it is enough to have $H^{(n)}(\omega,t)\to H(\omega,t)$ for all $t$ and all $\omega\in N^c$, for a single $\mathbb{P}$-negligible set $N$. Indeed, if we define a stopping time $\tau(\omega) = 0\{\omega\in N\} + \infty\{\omega\in N^c\}$ then

$$H^{(n)}((0,\tau]]\to H((0,\tau]]\qquad\text{pointwise}$$

and $\big(H^{(n)}((0,\tau]]\big)\bullet X_t = \{\omega\in N^c\}H^{(n)}\bullet X_t$, that is, $\big(H^{(n)}((0,\tau]]\big)\bullet X$ and $H^{(n)}\bullet X$ are indistinguishable. Similarly $(H((0,\tau]])\bullet X$ and $H\bullet X$ are indistinguishable. The uccp assertion of the Theorem implies the apparently stronger assertion.

Sometimes it is helpful to weaken the assumption of pointwise convergence even further. I believe I can show the following:
if $X \in \mathbb{SMG}_0$ and if $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ and $H^{(n)} \xrightarrow{ucpc} H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ then $H^{(n)} \bullet X \xrightarrow{ucpc} H \bullet X$. If I manage to fill in all the gaps in my proof I'll give it to you as an extra problem.

<6> **Example.** Suppose $\sigma$ and $\tau$ are stopping times and $X \in \mathbb{SMG}$. With $Y$ an $\mathcal{F}_\sigma$-measurable random variable, define $H = Y(\omega)((\sigma, \tau]]$. Show that $H \bullet X_t = Y(\omega)\,(X_{t\wedge\tau} - X_{t\wedge\sigma})$ by the following steps.

(i) Start with the case where $Y = F \in \mathcal{F}_\sigma$. Define new stopping times $\sigma' = \sigma F + \infty F^c$ and $\tau' := \tau F + \infty F^c$. Show that

$$(F((\sigma, \tau]]) \bullet X_t = X_{t\wedge\tau'} - X_{t\wedge\sigma'} = F\,(X_{t\wedge\tau} - X_{t\wedge\sigma})\,.$$

(ii) Extend the equality to all bounded, $\mathcal{F}_\sigma$-measurable $Y$ by a generating class argument.

(iii) For an unbounded $Y$, define stopping times

$$\tau_k = \sigma\{|Y| > k\} + \infty\{|Y| \le k\}.$$

Show that $((0, \tau_k]] = ((0, \sigma]]\{|Y| > k\} + \{|Y| \le k\}$. Deduce that the sequence $H^{(n)} := H((0, \tau_n]]$ is locally uniformly bounded and it converges pointwise to $H$.

(iv) Complete the argument.

The class of semimartingales is quite large. It is stable under sums (not surprising) and products (very surprising—see the next Section) and under exotic things like change of measure (to be discussed in a later Project). Even more surprisingly, semimartingales are the natural class of integrators for stochastic integrals; they are the unexpected final product of a long sequence of ad hoc improvements. You might consult Dellacherie (1980) or Protter (1990, pages 44; 87–88; 114), who expounded the whole theory by starting from plausible linearity and continuity assumptions then working towards the conclusion that only semimartingales can have the desired properties. See also the review by Protter (1986) of three books on stochastic integration.

## 8.3 Quadratic variation

In the proof of Lévy's martingale characterization of Brownian Motion, you saw how a sum of squares of increments of Brownian motion, taken over a partition via stopping times of an interval $[0, t]$, converges in probability to $t$. In fact, if one allows random limits, the behaviour is a general property of semimartingales. The limit is called the ***quadratic variation process*** of the semimartingale.

It is easiest to establish existence of the quadratic variation by means of an indirect stochastic integral argument. Suppose $X$ is an R-processes with $X_0 \equiv 0$. For each $t \in (0, \infty)$ define the left-limit process $X_t^{\ominus} := X(t-, \omega) := \lim_{s \uparrow\uparrow t} X(s, \omega)$. Problem [2] shows that $X^{\ominus} \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

Awkward and nonstandard notation, $X^{\ominus}$, but I want $X^-$ for the negative part of $X$.

<7> **Definition.** *The quadratic variation process of an $X$ in $\mathbb{SMG}_0$ is defined as $[X, X]_t := X_t^2 - 2X^{\ominus} \bullet X_t$ for $t \in \mathbb{R}^+$. For general $Z \in \mathbb{SMG}$, define $[Z, Z] := [X, X]$ where $X_t := Z_t - Z_0$.*

**Remark.** Some authors write $[X]$ instead of $[X, X]$.

The logic behind the name *quadratic variation* and one of the main reasons for why it is a useful process both appear in the next Theorem. The first assertion of the Theorem could even be used to define quadratic variation, but then we would have to work harder to prove existence of the limit (as for the quadratic variation of Brownian motion).

<8> **Definition.** *A **random grid** $\mathbb{G}$ is defined by a finite sequence of finite stopping times $0 \le \tau_0 \le \tau_1 \le \cdots \le \tau_k$. The mesh of the grid is defined as mesh$(\mathbb{G}) := \max_i |\tau_{i+1} - \tau_i|$; the max of the grid is defined as $\max(\mathbb{G}) := \tau_k$.*

To avoid double subscripting, let me write $\sum_{\mathbb{G}}$ to mean a sum taken over the stopping times that make up a grid $\mathbb{G}$.

<9> **Theorem.** *Suppose $X \in \mathbb{SMG}_0$ and $\{\mathbb{G}_n\}$ is a sequence of random grids with mesh$(\mathbb{G}_n) \xrightarrow{a.s.} 0$ and $\max(\mathbb{G}_n) \xrightarrow{a.s.} \infty$. Then:*

*(i)* $\sum_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)^2 \xrightarrow{ucpc} [X, X]_t$.

*(ii) The process $[X, X]$ has increasing sample paths;*

*(iii) If $\tau$ is a stopping time then $[X_{\wedge\tau}, X_{\wedge\tau}] = [X, X]_{\wedge\tau}$.*

Mention jumps as well?

Proof

(a) Consider first a fixed $t$ and a fixed grid $\mathbb{G}$: $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_k$. Define a left-continuous process $H_{\mathbb{G}} = \sum_{\mathbb{G}} X_{\tau_i}((\tau_i, \tau_{i+1}]]$. Use Example 6 to show that $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ and $H_{\mathbb{G}} \bullet X_t = \sum_{\mathbb{G}} X_{\tau_i}\left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)$.

(b) Except on a negligible set of paths (which I will ignore for the rest of the proof), show that $H^{(n)} := H_{\mathbb{G}_n}$ converges pointwise to the left-limit process $X^{\ominus}$ as mesh$(\mathbb{G}) \to 0$ and max$(\mathbb{G}) \to \infty$. Show also that $\{H^{(n)}\}$ is locally uniformly bounded. Hint: Consider stopping times $\sigma_k := \inf\{s : |X_s| \ge k\}$.

(c) Abuse notation by writing $\Delta_i X$ for $X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}$. Show that

$$\sum\nolimits_{\mathbb{G}_n} X_{\tau_i}(\Delta_i X) = H^{(n)} \bullet X_t \xrightarrow{ucpc} X^{\ominus} \bullet X_t$$

(d) Show that

$$2H^{(n)} \bullet X_t + \sum\nolimits_{\mathbb{G}_n} (\Delta_i X)^2 = X^2_{t\wedge\tau_k} \xrightarrow{ucpc} X_t^2.$$

(e) Complete the proof of (i).

(f) Establish (ii) by taking the limit along a sequence of grids (deterministic grids would suffice) for which both $s$ and $t$ are always grid points. Note: The sums of squared increments that converge to $[X, X]_t$ will always contain extra terms in addition to those for sums converging to $[X, X]_s$.

(g) For assertion (iii), merely note that $\tau \wedge t$ is one of the points in the interval $[0, t]$ over which the convergence in probability is uniform. Thus

$$\sum\nolimits_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}\wedge\tau} - X_{t\wedge\tau_i\wedge\tau}\right)^2 \xrightarrow{\mathbb{P}} [X, X]_{t\wedge\tau}.$$

Interpret the left-hand side as an approximating sum of squares for $[X_{\wedge\tau}, X_{\wedge\tau}]_t$.

□

Lect 17, Monday 22 March

<*10*> **Corollary.** *The square of a semimartingale $X$ is a semimartingale.*

Proof Let $Z_t := X_t - X_0 = M_t + A_t$, where $M_{\wedge\tau_k} \in \mathcal{M}_0^2(\mathbb{R}^+)$ for a localizing sequence $\{\tau_k\}$ and $A \in \mathbb{FV}_0$. Rearrange the definition of the square bracket process, $Z_t^2 = 2Z^\ominus \bullet Z_t + [Z,Z]_t$, to express $Z_t^2$ as a sum of a semimartingale and an increasing process. The process $X_t^2$ expands to $X_0^2 + Z_t^2 + 2X_0 M_t + 2X_0 A_t$. The last term belongs to $\mathbb{FV}_0$. The third term is reduced to $\mathcal{M}_0^2(\mathbb{R}^+)$ by the stopping times $\tau_k \wedge \sigma_k$, where $\sigma_k := 0\{|X_0| > k\} + \infty\{|X_0| \le k\}$.

□

<11> **Corollary.** *The product of two semimartingales is a semimartingale.*

Proof Use the ***polarization identity***, $4XY = (X+Y)^2 - (X-Y)^2$, and the fact that sums of semimartingales are semimartingales, to reduce the assertion to the previous Corollary.

□

<12> **Example.** The jump process $\Delta Y$ associated with an R-process $Y$ is defined by $\Delta_0 Y \equiv 0$ and $\Delta_t Y = Y_t - Y_t^\ominus$ for $0 < t < \infty$. For each $X$ in $\mathbb{SMG}_0$ show that $\Delta_t[X,X] = (\Delta_t X)^2$ (almost surely) by the following steps.

**Remark.** As a consequence, if a semimartingale $X$ has continuous sample paths then so does $[X,X]$.

(i) If $Y_n \xrightarrow{ucpc} Y$, show that $\Delta Y_n \xrightarrow{ucpc} \Delta Y$.

(ii) For each stopping time $\tau$, show that $\Delta_t(X_{\wedge\tau}) = \{t \le \tau\}\Delta_t X$

(iii) For the $H^{(n)}$ used in the proof of Theorem 9, show that $\Delta_t H^{(n)} \bullet X = (\Delta_t X) H_t^{(n)}$. Deduce that $\Delta_t(X^\ominus \bullet X) = (\Delta_t X) X_t^\ominus$ almost surely.

(iv) Show that $\Delta_t X^2 = (\Delta_t X)^2 + 2(\Delta_t X) X_t^\ominus$.

(v) Complete the argument.

□

<13> **Definition.** *For $X, Y \in \mathbb{SMG}_0$, the **square bracket process** $[X,Y]$ (also known as the quadratic covariation of the process of $X$ and $Y$) is defined, by polarization, as*

$$\begin{aligned}
4[X,Y]_t &:= [X+Y, X+Y]_t - [X-Y, X-Y]_t \\
&= (X_t + Y_t)^2 - (X_t - Y_t)^2 \\
&\quad - 2(X+Y)^\ominus \bullet (X+Y)_t + 2(X-Y)^\ominus \bullet (X-Y)_t \\
&= 4X_t Y_t - 4X^\ominus \bullet Y_t - 4Y^\ominus \bullet X_t.
\end{aligned}$$

**Remark.** Notice that $[X,Y]$ is equal to the quadratic variation process $[X,X]$ when $X \equiv Y$. Notice also that $[X,Y] \in \mathbb{FV}_0$, being a difference of two increasing processes started at 0.

The square bracket process inherits many properties from the quadratic variation. For example, you might prove that a polarization argument derives the following result from Theorem <9>.

<*14*> **Theorem.** *Let $X$ and $Y$ be semimartingales, and $\{\mathbb{G}_n\}$ be a sequence of random grids with mesh$(\mathbb{G}_n) \xrightarrow{a.s.} 0$ and $\max(\mathbb{G}_n) \xrightarrow{a.s.} \infty$. Then*

<15>
$$\sum\nolimits_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)\left(Y_{t\wedge\tau_{i+1}} - Y_{t\wedge\tau_i}\right) \xrightarrow{ucpc} [X,Y]_t,$$

*and $[X_{\wedge\tau}, Y_{\wedge\tau}] = [X_{\wedge\tau}, Y] = [X, Y_{\wedge\tau}] = [X,Y]_{\wedge\tau}$ for each stopping time $\tau$,*

## 8.4 Problems

[1] If $H$ and $K$ are in loc$\mathcal{H}_{\text{Bdd}}$, and $X$ is a semimartingale, show that $K \bullet (H \bullet X) = (KH) \bullet X$ for almost all paths. Hint: For fixed $H$, define $\mathcal{I}(K) := (HK) \bullet M$. What do you get when $K = ((0,\tau]]$?

[2] For each R-process $X$ indexed by $\mathbb{R}^+$, show that $X^\ominus \in$ loc$\mathcal{H}_{\text{Bdd}}$ by the following steps.

(i) Suppose $f$ is a cadlag function on $\mathbb{R}^+$. Show that $\sup_{0\le s\le t} |f(s)| < \infty$ for each $t$. Hint: Cover $[0,t]$ by finitely many intervals $(t_i - \delta_i, t_i + \delta_i)$ within which $\max\left(|f(s) - f(t_i-)|, |f(s) - f(t_i)|\right) < \epsilon$.

(ii) Show that $\sup_{0<s\le t} |X^\ominus(\omega,s)| < \infty$ for each $(\omega,t)$.

(iii) Show that $\tau_k(\omega) := \inf\{s \in \mathbb{R}^+ : |X^\ominus(\omega,s)| > k\}$ is a localizing sequence for $X^\ominus$.

[3] Suppose $M \in$ loc$\mathcal{M}_0^2(\mathbb{R}^+)$.

(i) Show that the process $X_t := M_t^2 - [M,M]_t$ belongs to loc$\mathcal{M}_0^2(\mathbb{R}^+)$.

(ii) Suppose $M$ has continuous sample paths and $[M,M]_t \equiv t$. Show that $M$ is a standard Brownian motion.

## References

Dellacherie, C. (1980). Un survol de la theorie de l'integrale stochastique. *Stochastic Processes and their Applications 10*(2), 115–144.

Dellacherie, C. and P. A. Meyer (1982). *Probabilities and Potential B: Theory of Martingales*. Amsterdam: North-Holland.

Protter, P. (1986). Review: Three books on stochastic integration. *Annals of Probability 14*(1), 343–346.

Protter, P. (1990). *Stochastic Integration and Differential Equations*. New York: Springer.

## Appendix D

# Functions of finite or bounded variation

Suppose $f$ is a real function defined on $\mathbb{R}^+$. For each finite grid

$$\mathbb{G}: \quad a = t_0 < t_1 < \cdots < t_N = b$$

on $[a, b]$ define the variation of $f$ over the grid to be

$$V_{f,\mathbb{G}}[a, b] := \sum\nolimits_{i=1}^{N} |f(t_i) - f(t_{i-1})|$$

Say that $f$ is of ***bounded variation*** on the interval $[a, b]$ if

$$V_f[a, b] := \sup\nolimits_{\mathbb{G}} V_{f,\mathbb{G}}[a, b] \text{ is finite}$$

where the supremum is taken over the set of all finite grids $\mathbb{G}$ on $[a, b]$. Say that $f$ is of ***finite variation*** if it is of bounded variation on each bounded interal $[0, b]$.

The key fact is: a function $f : \mathbb{R}^+ \to \mathbb{R}$ is of finite variation if and only if it can be written as a difference of two nondecreasing functions. You can establish this fact, and corresponding analogs for random processes, by the following steps.

(i) Suppose $f = f_1 - f_2$, where $f_1$ and $f_2$ are increasing functions on $\mathbb{R}^+$. Show that

$$V_f[0, b] \le V_{f_1}[0, b] + V_{f_2}[0, b] = f_1(b) - f_1(0) + f_2(b) - f_2(0).$$

Deduce that $f$ is of finite variation.

(ii) Suppose $f$ is a function on $\mathbb{R}^+$ with finite variation. Show that the functions $t \mapsto V_f[0, t]$ and $t \mapsto V_f[0, t] - f(t)$ are both nondecreasing and

$$f(t) = V_f[0, t] - \big(V_f[0, t] - f(t)\big),$$

a difference of two nondecreasing functions. In what follows, drop the subscript $f$ on the variation functions.

(a) Suppose $\mathbb{G}$ is a grid on $[a, b]$ and that $s$ is point of $(a, b)$ that is not already a grid point. Show that $V(\mathbb{G}, [a, b])$ is increased if we add $s$ as a new grid point.

(b) Show that $V[0,a] + V[a,b] = V[0,b]$ for all $a < b$. Deduce that $t \mapsto V[0,t]$ is an increasing function

(c) Suppose $0 < s < t$. Show that

$$V[0,t]-f(t) = V[0,s]-f(s)+f(s)-f(t)+V[s,t] \geq V[0,s]-f(s).$$

Hint: Consider a two-point grid on $[s,t]$.

(iii) Now suppose $f$ is not only of finite variation but is also right-continuous at some $a \in \mathbb{R}^+$. For a fixed $b > a$ and an $\epsilon > 0$ choose a grid

$$\mathbb{G}: \quad a = t_0 < t_1 < \cdots < t_N = b$$

for which $V(\mathbb{G},[a,b]) > V[a,b] - \epsilon$. With no loss of generality suppose $|f(t_1) - f(a)| < \epsilon$. Show that

$$\epsilon + V[t_1,b] \geq V(\mathbb{G},[a,b]) > V[a,t_1] + V[t_1,b] - \epsilon$$

Deduce that $t \mapsto V[0,t]$ is continuous from the right at $a$.

(iv) If $f$ is right-continuous everywhere, show that each $V_f[a,b]$ can be determined by taking a supremum over equispaced grids on $[a,b]$.

(v) If $X$ is an R-processes with sample paths of finite variation, show that it can be expressed as the difference of two R-processes with increasing sample paths. [The issue is whether $V_{X(\cdot,\omega)}[0,t]$ is adapted.]