Project 9

The Itô formulae

This Project will derive the Itô formula only for semimartingales with continuous sample paths. Those of you who want to understand the formula for processes with jumps should consult Dellacherie and Meyer (1982, §VIII.24–28) or Protter (1990, page 71).

9.1 Itô formulae

Let f be a continuous, real-valued function defined on some open subset G of $\mathbb{R}^2$. Suppose f has two continuous partial derivatives f_x and f_{xx} with respect to its first argument and a continuous partial derivative f_y with respect to its second argument. For a process $\{(X_t, Y_t) : t \in \mathbb{R}^+\}$ taking values in G, define new processes by

<1>
$$\begin{aligned} F_x(\omega, s) &:= f_x\big(X(\omega, s), Y(\omega, s)\big), \\ F_{xx}(\omega, s) &:= f_{xx}\big(X(\omega, s), Y(\omega, s)\big) \\ F_y(\omega, s) &:= f_y\big(X(\omega, s), Y(\omega, s)\big). \end{aligned}$$

Each of them is adapted and has continuous paths; each process is predictable.

<2> **Theorem.** ***[Itô Formula]*** *Suppose X and Y are semimartingales with continuous sample paths, such that the two-dimensional random process $\{(X_t, Y_t) : t \in \mathbb{R}^+\}$ takes values in an open subset G of $\mathbb{R}^2$. Suppose Y has paths of bounded variation. For f as described above and processes F_x, F_y, and F_{xx} as defined in <1>, the process $f(X_s, Y_s)$ is a semimartingale with*

$$f(X_t, Y_t) - f(X_0, Y_0) = F_x \bullet X_t + \tfrac{1}{2} F_{xx} \bullet [X, X]_t + F_y \bullet Y_t$$

for each t in $\mathbb{R}^+$.

Remark. The Itô formula is often written in the suggestive form

$$df(X_t, Y_t) = f_x(X_t, Y_t)\, dX_t + \tfrac{1}{2} f_{xx}(X_t, Y_t)\, d[X, X]_t + f_y(X_t, Y_t)\, dY_t,$$

which hints at its origins as a sum of small increments.

The process $\frac{1}{2} F_{xx} \bullet [X, X] + F_y \bullet Y$ is in $\mathbb{FV}$. If $X \in \text{loc}\mathcal{M}^2(\mathbb{R}^+)$ then $F_x \bullet X \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. The Itô formula then gives the semimartingale decomposition for the process $f(X_t, Y_t)$.

Proof Let K be a compact subset of G. Define

$$\sigma := \inf\{t \in \mathbb{R}^+ : (X_t, Y_t) \notin K\}.$$

Replace X and Y by the corresponding stopped processes $X_{\wedge\sigma}$ and $Y_{\wedge\sigma}$.

> **Remark.** For the steps that follow I haven't been very careful about distinguishing between process W for which $W_0 \equiv 0$ and those that might start somewhere else. I don't think it matters much, but you should be extra careful in checking my assertions.

(i) Show that the formula is trivially true for the stopped processes if $(X_0, Y_0) \notin K$.

(ii) Suppose g is a continuous real function defined on G. For each $\epsilon > 0$ show that there exists a $\delta > 0$ for which:

$$|g(x+\Delta x, y+\Delta y) - g(x,y)| \le \epsilon \qquad \text{if } (x,y) \in K \text{ and } \max(|\Delta x|, |\Delta y|) \le \delta.$$

> **Remark.** If $(x+\Delta x, y+\Delta y) \in K$, these properties are just a statement of the uniform continuity for the restriction of g to K; if $(x+\Delta x, y+\Delta y)$ is allowed to poke outside K, the argument is only a tiny bit more subtle. I believe the result as stated is needed if we consider Taylor expansion along line segments between points in K if K is not convex.

(iii) For $\max(|\Delta x|, |\Delta y|) \le \delta$ and $(x,y) \in K$, show that

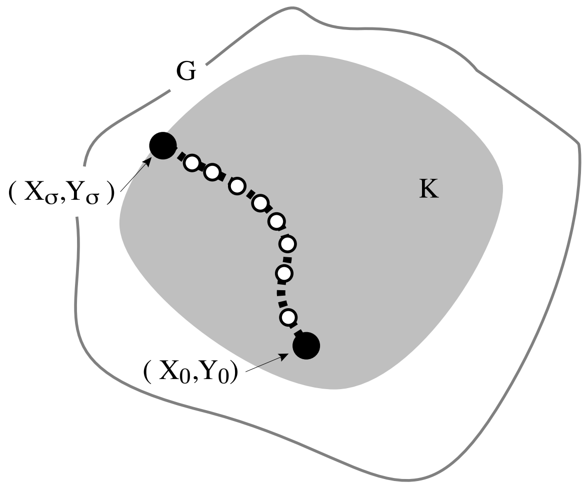
$$\begin{aligned} &f(x+\Delta x, y+\Delta y) - f(x,y) \\ &\qquad = (\Delta x) f_x(x,y) + \tfrac{1}{2}(\Delta x)^2 f_{xx}(x,y) + (\Delta y) f_y(x,y) + \text{REM} \\ &\qquad\qquad \text{where } \text{REM} \le \epsilon\big(\tfrac{1}{2}(\Delta x)^2 + |\Delta y|\big) \end{aligned}$$

Hint: First consider the representation $g(1) - g(0) = \int_0^1 g'(s)\,ds$ for the function $g(s) := f(x+\Delta x, y+s\Delta y) - f(x+\Delta x, y) - \Delta y f_y(x,y)$. Then argue similarly for $f(x+s\Delta x, y)$.

(iv) Fix t. Consider a sequence $\epsilon_n \downarrow 0$. Let δ_n be a sequence for which (ii) holds with g equal to any of fx, f_{xx}, or f_y. Define a grid $\mathbb{G}_n$ via stopping times

$$\tau_{i+1} := \inf\{s \ge \tau_i : |(X,Y)_s - (X,Y)_{\tau_i}| \ge \delta_n\} \wedge t \wedge \sigma.$$

Show that there exist integers $k(n)$ such that $\mathbb{P}\{\tau_{k(n)} = t \wedge \sigma\} \to 1$ as $n \to \infty$.



(v) Write $\Delta_i X$ for $X_{\tau_{i+1}} - X_{\tau_i}$, and similarly for Y. Show that the increment $f(X_{\tau_{k(n)}}, Y_{\tau_{k(n)}}) - f(X_0, Y_0)$ differs from

<3> $$\sum\nolimits_{i=0}^{k(n)-1} (\Delta_i X)F_x(\tau_i) + \tfrac{1}{2}(\Delta_i X)^2 F_{xx}(\tau_i) + (\Delta_i Y)F_y(\tau_i)$$

by a quantity that tends in probability to zero.

(vi) Show that the contribution from the first summand in <3> equals $H^{(n)} \bullet X_{t\wedge\tau_{k(n)}}$, where

$$H^{(n)}(\omega, s) := \sum\nolimits_{i=0}^{k(n)-1} F_x(\tau_i, \omega)((\tau_i, \tau_{i+1}]].$$

Show that $H^{(n)} - F_x((0, \tau_{k(n)}]]$ is uniformly bounded and converges pointwise to F_x.

(vii) Deduce that $\sum_{i=0}^{k(n)-1}(\Delta_i X)F_x(\tau_i) \xrightarrow{\mathbb{P}} F_x \bullet X_{t\wedge\sigma}$.

(viii) Argue similarly for the contribution from the third summand in <3>.

(ix) The argument (Protter 1990, page 69) for the second summand in <3> is a little more complicated. Define $Z_t := X_t - X_0$ and write $\Delta_i Z$ for $Z_{\tau_{i+1}} - Z_{\tau_i}$.

(a) Show that

$$\sum\nolimits_{i=0}^{k(n)-1} (\Delta_i X)^2 F_{xx}(\tau_i) = \sum\nolimits_{i=0}^{k(n)-1} F_{xx}(\tau_i)(Z^2_{\tau_{i+1}} - Z^2_{\tau_i}) - 2\sum\nolimits_{i=0}^{k(n)-1} \left(F_{xx}(\tau_i) Z_{\tau_i}\right)(\Delta_i Z).$$

(b) Show that the right-hand side converges in probability to

$$\begin{aligned} F_{xx} \bullet Z^2_{t\wedge\sigma} &- 2(F_{xx}Z) \bullet Z_{t\wedge\sigma} \\ &= F_{xx} \bullet (Z^2 - 2Z \bullet Z)_{t\wedge\sigma} \\ &= F_{xx} \bullet [Z,Z]_{t\wedge\sigma} = F_{xx} \bullet [X,X]_{t\wedge\sigma}. \end{aligned}$$

(x) Deduce that

$$\begin{aligned} f(X_{t\wedge\sigma}, Y_{t\wedge\sigma}) &- f(X_0, Y_0) \\ &= F_x \bullet X_{\wedge\sigma t} + \tfrac{1}{2} F_{xx} \bullet [X_{\wedge\sigma}, X_{\wedge\sigma}]_t + F_y \bullet Y_{\wedge\sigma t} \\ &= F_x \bullet X_{t\wedge\sigma} + \tfrac{1}{2} F_{xx} \bullet [X,X]_{t\wedge\sigma} + F_y \bullet Y_{t\wedge\sigma}. \end{aligned}$$

(xi) Complete the proof by letting K expand up to G, so that $\sigma \uparrow \infty$.

□

<4> **Example.** Let $\{X_t : t \in \mathbb{R}^+\}$ be a locally square integrable martingale with continuous sample paths. Its quadratic variation process $Y := [X,X]$ is continuous and of bounded variation. To be on the safe side, let me also assume that $X_0 \equiv 0$, even though it is not necessary.

The semimartingale $Z_t := \exp(X_t - \frac{1}{2}Y_t)$ is a candidate for an application of the Itô formula, with $f(x,y) = \exp(x - \frac{1}{2}y)$. We have $F_x = F_{xx} = -2F_y = Z$, and

$$Z_t - Z_0 = Z \bullet X + \tfrac{1}{2} Z \bullet [X,X]_t - \tfrac{1}{2} Z \bullet Y_t = Z \bullet X_t.$$

The Z process is also a locally square integrable martingale with continuous paths.

□

<5> **Example.** Let X be a locally square integrable martingale with continuous sample paths, $X_0 \equiv 0$, and for which $[X,X]_t = t$. For a fixed real θ, define $Z_t := \exp(i\theta X_t + \frac{1}{2}\theta^2 t)$. Apply the Itô formula (to real and imaginary parts) to show that $Z_t = 1 + i\theta Z \bullet X_t$. The Z process is also a locally square integrable martingale with continuous paths. For some localizing sequence of stopping times $\{\tau_k\}$,

$$\mathbb{P} Z_{t\wedge\tau_k} = 1 + i\theta \mathbb{P} Z \bullet X_{t\wedge\tau_k} = 1$$

A dominated convergence argument then shows that $\mathbb{P}\exp(i\theta X_t) = \exp(-\frac{1}{2}\theta^2 t)$ for every real θ. Thus $X_t \sim N(0,t)$.

With a little more work you should be able to extend the preceding argument to establish Lévy's characterization of Brownian Motion, in a slightly more general form than the one discussed in Project 5.

□

There would be nothing to gain in Theorem 2 by requiring existence of second-order partial derivatives f_{xy} and f_{yy}: the corresponding bracket process $[X,Y]$ and $[Y,Y]$ are both zero, because the process Y has continuous paths of finite variation. The story would change if Y did not have paths of bounded variation. In that case, the error term $\epsilon_n \sum_i |\Delta_i Y|$ would no longer disappear in the limit. We would instead need continuous second order partial derivatives f_{xy} and f_{yy} to handle the contributions from the $\Delta_i Y$ increments to the Taylor expansion (to quadratic terms) in both variables. Error terms like

$$\epsilon_n \sum\nolimits_i (\Delta_i Y)^2 + (\Delta_i X)(\Delta_i Y)$$

would again converge in probability to zero. The cross-product term

$$\begin{aligned}
&\sum\nolimits_i F_{xy}(\tau_i)(\Delta_i X)(\Delta_i Y) \\
&\qquad = \sum\nolimits_i F_{xy}(\tau_i)(X_{\tau_{i+1}} Y_{\tau_{i+1}} - X_{\tau_i} Y_{\tau_i}) \\
&\qquad\quad - \sum\nolimits_i F_{xy}(\tau_i)\left(X_{\tau_i}(\Delta_i Y) + Y_{\tau_i}(\Delta_i X)\right)
\end{aligned}$$

would converge in probability to

$$F_{xy} \bullet (XY - X_0 Y_0 - X \bullet Y - Y \bullet X)_t = F_{xy} \bullet [X,Y]_t.$$

A similar argument works for functions of more than two semimartingales.

<6> **Theorem.** ***[Multiprocess Itô Formula]*** *Suppose $\mathbf{X} = (X^{(1)}, \dots X^{(d)})$ and $\mathbf{Y} = (Y^{(1)}, \dots Y^{(d')})$ are semimartingales with continuous paths, such that the $d + d'$-dimensional random process $(\mathbf{X}, \mathbf{Y})$ takes values in an open subset G of $\mathbb{R}^{d+d'}$. Suppose each $Y^{(\gamma)}$ has paths of finite variation.*

If f is a continuous, real-valued function on G with continuous partial derivatives $f_{x(\alpha)}$, $f_{x(\alpha),x(\beta)}$, $f_{y(\gamma)}$ for $\alpha, \beta = 1, \dots, d$ and $\gamma = 1, \dots, d'$, then $f(\mathbf{X}, \mathbf{Y})$ is a semimartingale with

$$\begin{aligned}
&f(\mathbf{X}_t, \mathbf{Y}_t) - f(\mathbf{X}_0, \mathbf{Y}_0) \\
&\qquad = \sum\nolimits_\alpha F_{x(\alpha)} \bullet X_t^{(\alpha)} + \sum\nolimits_\gamma F_{y(\gamma)} \bullet Y_t^{(\gamma)} \\
&\qquad\quad + \tfrac{1}{2} \sum\nolimits_{\alpha,\beta} F_{x(\alpha),x(\beta)} \bullet [X^{(\alpha)}, X^{(\beta)}]_t
\end{aligned}$$

for each t in $\mathbb{R}^+$.

References

Dellacherie, C. and P. A. Meyer (1982). *Probabilities and Potential B: Theory of Martingales*. Amsterdam: North-Holland.

Protter, P. (1990). *Stochastic Integration and Differential Equations*. New York: Springer.