FKG and related inequalities

For \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\) define \(x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n)\) and \(x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)\).

Write \(x \leq y\) to mean \(x \vee y = x\). Say that a function \(f\) on \(\mathbb{R}^n\) is increasing if it is an increasing function in each of its arguments (for fixed values of the other arguments). Equivalently, \(f\) is increasing if \(f(x) \leq f(y)\) whenever \(x \leq y\).

\(<1>\) **Theorem.** Suppose \(P\) and \(Q\) are probability measures on \(\mathcal{B}(\mathbb{X}^n)\) with densities \(p = dP/d\mu\) and \(q = dQ/d\mu\) with respect to a product measure \(\mu\). Suppose

\[
p(x)q(y) \leq p(x \wedge y)q(x \vee y) \quad \text{for all } x, y \in \mathbb{X}^n
\]

Then \(Pf \leq Qf\) for each increasing function \(f\) that is both \(P\)- and \(Q\)-integrable.

\(<2>\) **Theorem.** Suppose \(P\) is a probability measure with a density \(p = dP/d\mu\) with respect to a product measure \(\mu\), for which

\[
p(x)p(y) \leq p(x \wedge y)p(x \vee y) \quad \text{for all } x, y \in \mathbb{X}^n
\]

If \(f\) and \(g\) are increasing, \(P\)-square integrable functions on \(\mathbb{X}^n\) then \(Pf(x)g(x) \geq (Pf)(Pg)\). That is, \(f\) and \(g\) are positively correlated as random variables under \(P\).

Both results will follow as special case of the following general inequality.

\(<3>\) **Theorem.** Suppose \(f_1, \ldots, f_4\) are nonnegative, Borel-measurable functions on \(\mathbb{X}^n\), where \(X \subseteq \mathbb{R}\), for which

\[
f_1(x)f_2(y) \leq f_3(x \wedge y)f_4(x \vee y) \quad \text{for all } x, y \in \mathbb{X}^n
\]

Let \(\mu = \mu_1 \otimes \ldots \otimes \mu_n\) be a sigma-finite product measure on \(\mathcal{B}(\mathbb{X}^n)\). Then

\[
\mu(f_1)\mu(f_2) \leq \mu(f_3)\mu(f_4)
\]

**Proof.** The method of proof is induction on \(n\). The main idea is that an inequality analogous to \(<4>\) is preserved by integration over a single coordinate.

To make the patterns easier to see, I adopt some temporary notation that focuses attention on the \(n\)th coordinate. Write \(x = (X, u)\) and \(y = (Y, v)\), where \(X = (x_1, \ldots, x_{n-1})\) and \(Y = (y_1, \ldots, y_{n-1})\), and let \(M = \mu_n\). Inequality \(<4>\) then becomes

\(<5>\)

\[
f_1(X, u)f_2(Y, v) \leq f_3(X \wedge Y, u \wedge v)f_4(X \vee Y, u \vee v) \quad \text{for all } X, Y, u, v.
\]

We need to show that

\(<6>\)

\[
\tilde{f}_1(X)\tilde{f}_2(Y) \leq \tilde{f}_3(X \wedge Y)\tilde{f}_4(X \vee Y) \quad \text{where } \tilde{f}_i(Z) := M^n f_i(Z, w).
\]

The arguments \(X\) and \(Y\) stay fixed throughout the inductive step. The important calculations all involve the functions

\[
A(u, v) := f_1(X, u)f_2(Y, v) = B(v, u)
\]

\[
C(u, v) := f_3(X \wedge Y, u)f_4(X \vee Y, v) = D(v, u)
\]

Replacement of the dummy variables \((u, v)\) by \((v', u')\) on the set \([u < v]\) transforms the left-hand side of \(<6>\) to

\[
M^n M^n [u = v]f_1(X, u)f_2(Y, v) \\
+M^n M^n [u < v]f_1(X, u)f_2(Y, v) \\
+M^n M^n [v' > u']f_1(X, v')f_2(Y, u') \\
= M^n M^n [u = v]A(u, v) + M^n M^n [u < v] (A(u, v) + B(u, v)).
\]
Similarly, the right-hand side of <6> equals
\[ M^u M^v [u = v] C(u, v) + M^u M^v [u < v] (C(u, v) + D(u, v)) \].

Inequality <5> gives \( A(u, v) \leq C(u, v) \) on the set \( \{ u = v \} \). On the set \( \{ u < v \} \) it gives both \( \max(A, B) \leq C \) and
\[
AB = f_1(X, u) f_2(Y, u) f_1(X, v) f_2(Y, v) \\
\leq f_3(X \land Y, u) f_4(X \lor Y, u) f_3(X \land Y, v) f_4(X \lor Y, v) = C D,
\]
which imply
\[
A(u, v) + B(u, v) \leq C(u, v) + D(u, v) \quad \text{on} \quad \{ u < v \}
\]
because
\[
0 \leq (1 - A/C)(1 - B/C) = 1 - (A + B)/C + (AB)/C^2 \\
\leq (C - A - B + D)/C.
\]

Multiple appeals to the inductive argument eventually reduce the assertion of the Theorem to the case where \( n = 1 \), which can be handled as in the argument leading to <6> with the extra simplification that there are no longer any \( X \) or \( Y \).

\[ \square \]

Proof of Theorem <1>. First consider the case where \( f \) is bounded and nonnegative. Define
\[
f_1(x) = p(x) f(x) \quad \text{and} \quad f_3(w) = p(w) \\
f_2(y) = q(y) \quad \text{and} \quad f_4(z) = q(z) f(z)
\]
Check that
\[
f_1(x) f_2(y) = f(x) p(x) q(y) \\
\leq f(x \land y) p(x \land y) q(x \lor y) = f_3(x \land y) f_4(x \lor y).
\]
Invoke Theorem <3>.

For the general case, apply the result just established to the function \( f_n := 2n \land (f + n)^+ \) to deduce that \( P(f_n - n) \leq Q(f_n - n) \). The let \( n \) tend
to infinity, noting that \( |f_n - n| \leq f \) and \( f_n - n \rightarrow f \) to justify an appeal to

Dominated Convergence.

\[ \square \]

Proof of Theorem <2>. As for the proof of Theorem <1>, it is enough to consider the case of bounded \( f \) and \( g \). Define
\[
f_1(x) = p(x) f(x) \quad \text{and} \quad f_3(w) = p(w) \\
f_2(y) = p(y) g(y) \quad \text{and} \quad f_4(z) = p(z) f(z) g(z)
\]
Check that
\[
f_1(x) f_2(y) = f(x) g(y) p(x) p(y) \\
\leq f(x \lor y) g(x \lor y) p(x \lor y) p(x \lor y) = f_3(x \lor y) f_4(x \lor y).
\]
Invoke Theorem <3>.

1. Notes

Notes unedited; might be incorrect

Theorem <3> is due to Ahlswede & Daykin (1978), but the proof comes from Karlin & Rinott (1980). Eaton (1986, Chapter 5) contains a nice exposition.
The original paper of Fortuin, Kasteleyn & Ginibre (1971) stated the result of Theorem <2> for increasing functions defined on a finite distributive lattice. It also contained applications to Physics, including the Ising model.

Preston (1974a) noted that finite distributive lattices can always be represented as a collection of subsets of some finite set. Equivalently, the points of such a lattice can be represented as $n$-tuples of 0’s and 1’s, or as $n$-tuples of ±1’s. Preston (1974b, Chapter 3) reproduced a proof Holley (1974), which was expressed as a coupling of two probability measures satisfying the setwise analog of the condition in Theorem <1>. In fact, a general coupling result of Strassen (1965) shows that the Holley result is equivalent to the result asserted by Theorem <1>.

See the survey by Den Hollander & Keane (1986) for more about the history of the FKG inequality and its variants.

REFERENCES


I need to get the history of FKG straight.