FKG and related inequalities

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ define

$$x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n) \qquad \text{and} \qquad x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n).$$

Write $x \leq y$ to mean $x \vee y = x$. Say that a function f on $\mathbb{R}^n$ is increasing if it is an increasing function in each of its arguments (for fixed values of the other arguments). Equivalently, f is increasing if $f(x) \leq f(y)$ whenever $x \leq y$.

<1> **Theorem.** *Suppose P and Q are probability measures on $\mathcal{B}(\mathcal{X}^n)$ with densities $p = dP/d\mu$ and $q = dQ/d\mu$ with respect to a product measure μ. Suppose*

$$p(x)q(y) \leq p(x \wedge y)q(x \vee y) \qquad \text{for all } x, y \in \mathcal{X}^n$$

Then $Pf \leq Qf$ for each increasing function f that is both P- and Q-integrable.

<2> **Theorem.** *Suppose P is a probability measure with a density $p = dP/d\mu$ with respect to a product measure μ, for which*

$$p(x)p(y) \leq p(x \wedge y)p(x \vee y) \qquad \text{for all } x, y \in \mathcal{X}^n$$

If f and g are increasing, P-square integrable functions on $\mathcal{X}^n$ then $Pf(x)g(x) \geq (Pf)(Pg)$. That is, f and g are positively correlated as random variables under P.

Both results will follow as special case of the following general inequality.

<3> **Theorem.** *Suppose $f_1, \ldots, f_4$ are nonnegative, Borel-measurable functions on $\mathcal{X}^n$, where $\mathcal{X} \subseteq \mathbb{R}$, for which*

<4>
$$f_1(x)f_2(y) \leq f_3(x \wedge y)f_4(x \vee y) \qquad \text{for all } x, y \in \mathcal{X}^n.$$

Let $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ be a sigma-finite product measure on $\mathcal{B}(\mathcal{X}^n)$. Then

$$\mu(f_1)\mu(f_2) \leq \mu(f_3)\mu(f_4)$$

Proof. The method of proof is induction on n. The main idea is that an inequality analogous to <4> is preserved by integration over a single coordinate.

To make the patterns easier to see, I adopt some temporary notation that focuses attention on the nth coordinate. Write $x = (X, u)$ and $y = (Y, v)$, where $X = (x_1, \ldots, x_{n-1})$ and $Y = (y_1, \ldots, y_{n-1})$, and let $M = \mu_n$. Inequality <4> then becomes

<5>
$$f_1(X, u)f_2(Y, v) \leq f_3(X \wedge Y, u \wedge v)f_4(X \vee Y, u \vee v) \qquad \text{for all } X, Y, u, v.$$

We need to show that

<6>
$$\widetilde{f_1}(X)\widetilde{f_2}(Y) \leq \widetilde{f_3}(X \wedge Y)\widetilde{f_4}(X \vee Y) \qquad \text{where } \widetilde{f_i}(Z) := M^w f_i(Z, w).$$

The arguments X and Y stay fixed throughout the inductive step. The important calculations all involve the functions

$$\begin{aligned} A(u, v) &:= f_1(X, u)f_2(Y, v) = B(v, u) \\ C(u, v) &:= f_3(X \wedge Y, u)f_4(X \vee Y, v) = D(v, u) \end{aligned}$$

Replacement of the dummy variables (u, v) by (v', u') on the set $\{u < v\}$ transforms the left-hand side of <6> to

$$\begin{aligned} &M^u M^v\{u = v\}f_1(X, u)f_2(Y, v) \\ &+M^u M^v\{u < v\}f_1(X, u)f_2(Y, v) \\ &+M^{v'} M^{u'}\{v' > u'\}f_1(X, v')f_2(Y, u') \\ &= M^u M^v\{u = v\}A(u, v) + M^u M^v\{u < v\}\left(A(u, v) + B(u, v)\right). \end{aligned}$$

Similarly, the right-hand side of <6> equals

$$M^u M^v\{u = v\}C(u, v) + M^u M^v\{u < v\}\big(C(u, v) + D(u, v)\big).$$

Inequality <5> gives $A(u, v) \le C(u, v)$ on the set $\{u = v\}$. On the set $\{u < v\}$ it gives both $\max(A, B) \le C$ and

$$\begin{aligned} AB &= f_1(X, u) f_2(Y, u) f_1(X, v) f_2(Y, v) \\ &\le f_3(X \wedge Y, u) f_4(X \vee Y, u) f_3(X \wedge Y, v) f_4(X \vee Y, v) = CD, \end{aligned}$$

which imply

$$A(u, v) + B(u, v) \le C(u, v) + D(u, v) \qquad \text{on } \{u < v\}$$

because

$$\begin{aligned} 0 \le (1 - A/C)(1 - B/C) &= 1 - (A + B)/C + (AB)/C^2 \\ &\le (C - A - B + D)/C. \end{aligned}$$

Multiple appeals to the inductive argument eventually reduce the assertion of the Theorem to the case where $n = 1$, which can be handled as in the argument leading to <6> with the extra simplification that there are no longer any X or Y. □

Proof of Theorem <1>. First consider the case where f is bounded and nonegative. Define

$$f_1(x) = p(x) f(x) \qquad \text{and} \qquad f_3(w) = p(w)$$
$$f_2(y) = q(y) \qquad \text{and} \qquad f_4(z) = q(z) f(z)$$

Check that

$$\begin{aligned} f_1(x) f_2(y) &= f(x) p(x) q(y) \\ &\le f(x \vee y) p(x \wedge y) q(x \vee y) = f_3(x \wedge y) f_4(x \vee y). \end{aligned}$$

Invoke Theorem <3>.

For the general case, apply the result just established to the function $f_n := 2n \wedge (f + n)^+$ to deduce that $P(f_n - n) \le Q(f_n - n)$. The let n tend to infinity, noting that $|f_n - n| \le f$ and $f_n - n \to f$ to justify an appeal to Dominated Convergence. □

Proof of Theorem <2>. As for the proof of Theorem <1>, it is enouugh to consider the case of bounded f and g. Define

$$f_1(x) = p(x) f(x) \qquad \text{and} \qquad f_3(w) = p(w)$$
$$f_2(y) = p(y) g(y) \qquad \text{and} \qquad f_4(z) = p(z) f(z) g(z)$$

Check that

$$\begin{aligned} f_1(x) f_2(y) &= f(x) g(y) p(x) p(y) \\ &\le f(x \vee y) g(x \vee y) p(x \wedge y) p(x \vee y) = f_3(x \wedge y) f_4(x \vee y). \end{aligned}$$

Invoke Theorem <3>. □

1. Notes

Notes unedited; might be incorrect

Theorem <3> is due to Ahlswede & Daykin (1978), but the proof comes from Karlin & Rinott (1980). Eaton (1986, Chapter 5) contains a nice exposition.

The original paper of Fortuin, Kasteleyn & Ginibre (1971) stated the result of Theorem <2> for increasing functions defined on a finite distributive lattice. It also contained applications to Physics, including the Ising model.

Preston (1974*a*) noted that finite distributive lattices can always be represented as a collection ofsubsets of some finite set. Equivalently, the points of such a lattice can be represented as n-tuples of 0's and 1's, or as n-tuples of ±1's. Preston (1974*b*, Chapter 3) reproduced a proof Holley (1974), which was expressed as a coupling of two probability measures satisfying the setwise analog of the condition in Theorem <1>. In fact, a general coupling result of Strassen (1965) shows that the Holley result is equivalent to the result asserted by Theorem <1>.

See the survey by Den Hollander & Keane (1986) for more about the history of the FKG inequality and its variants.

I need to get the history of FKG straight.

References

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