Chapter

Griffiths's inequalities

Let λ denote the uniform distribution on the product space $\Omega = \{-1, +1\}^S$, for some finite set S. For each subset a of S define $\phi_a(\omega) = \phi_a(\omega_a) = \prod_{i \in a} \omega_i$. When a is the empty set, the corresponding ϕ_a is identically 1. Note that $\phi_a(\omega)\phi_b(\omega) = \phi_{a\Delta b}(\omega)$, where $a\Delta b$ denotes the symmetric difference of a and b.

Let $\mathbb{F}$ denote the set of all nonempty subsets of S. For each t in $\mathbb{R}^{\mathbb{F}}$ define

$$H(\omega, t) = \sum\nolimits_{a \in \mathbb{F}} t_a \phi_a(\omega).$$

Define probability measures $\mathbb{P}_t$, for $t \in \mathbb{R}^{\mathbb{F}}$, by

$$\frac{d\mathbb{P}_t}{d\lambda} = \frac{\exp\big(H(\omega, t)\big)}{M(t)} \qquad \text{where } M(t) := \lambda \exp\big(H(\omega, t)\big)$$

Recall (see Pollard 2001, Problem C.2, for example) that the function $L(t) = \log M(t)$ is convex, with

$$\frac{\partial}{\partial t_a} L(t) = \mathbb{P}_t \phi_a(\omega)$$

$$\frac{\partial^2}{\partial t_b \partial t_a} L(t) = \text{cov}_t(\phi_a, \phi_b) = \mathbb{P}_t\left(\phi_a(\omega)\phi_b(\omega)\right) - \mathbb{P}_t\phi_a(\omega)\mathbb{P}\phi_b(\omega)$$

Consequently, $\partial \mathbb{P}_t \phi_a / \partial t_b = \text{cov}_t(\phi_a, \phi_b)$.

<1> **Theorem.** *If $t_a \geq 0$ for all $a \in \mathbb{F}$ then*

(i) $\mathbb{P}_t \phi_a \geq 0$ for all $a \in \mathbb{F}$

(ii) for all $a, b \in \mathbb{F}$,

$$\frac{\partial}{\partial t_b} \mathbb{P}_t \phi_a = \text{cov}_t(\phi_a, \phi_b) \geq 0$$

Proof. For all sets of integers $\{k_a : a \in \mathbb{F}\}$, note that

<2>
$$\lambda \prod_{a \in \mathbb{F}} \phi_a(\omega_a)^{k_a} = \lambda \prod_{i \in S} \omega_i^{\ell_i} = \begin{cases} 1 & \text{if } \ell_i \text{ is even for all } i \\ 0 & \text{otherwise} \end{cases},$$

where $\ell_i = \sum_a \{i \in a\} k_a$.

For (i), note that

$$M(t)\mathbb{P}_t\phi_a = \sum_{n=0}^{\infty} \lambda \phi_a(\omega) \frac{H(\omega, t)^n}{n!}$$

The nth summand expands to a sum of terms like

$$\frac{n!}{\prod_{b \in \mathbb{F}} k_b!} \lambda \phi_a \prod_{b \in \mathbb{F}} t_b^{k_b} \phi_b(\omega)^{k_b}$$

where the k_b are nonnegative integers summing to n. By <2>, each such summand is nonnegative if all t_b are nonnegative.

For (ii), first note that, for every function f on $\Omega \times \Omega$,

$$\lambda^{\omega}\lambda^{\omega'} f(\omega, \omega') = \lambda^{\omega}\lambda^{\omega'} f(\omega, \widetilde{\omega}) \qquad \text{where } \widetilde{\omega}_i := \omega_i \omega'_i \text{ for each } i$$

because $\widetilde{\omega}$ is uniformly distributed on Ω independently of ω under $\lambda \otimes \lambda$. Note also that $\phi_d(\widetilde{\omega}) = \phi_d(\omega)\phi_d(\omega')$ for all $d \in \mathbb{F}$. Thus

$$\begin{aligned} &M(t)^2 \operatorname{cov}_t(\phi_a, \phi_b) \\ &\quad = \lambda^{\omega}\lambda^{\omega'} \big(\phi_a(\omega)\phi_b(\omega) - \phi_a(\omega)\phi_b(\widetilde{\omega})\big) \exp\big(H(\omega, t) + H(\widetilde{\omega}, t)\big) \\ &\quad = \lambda^{\omega}\lambda^{\omega'} \phi_a(\omega)\phi_b(\omega)[1 - \phi_b(\omega')] \exp\left(\sum_{d \in \mathbb{F}} t_d \phi_d(\omega)[1 + \phi_d(\omega')]\right) \\ &\quad = \lambda^{\omega'}[1 - \phi_b(\omega')]\lambda^{\omega}\phi_{a\Delta b}(\omega) \exp\big(H(\omega, \tilde{t}\,)\big) \qquad \text{where } \tilde{t}_d = t_d[1 + \phi_d(\omega')]. \end{aligned}$$

Note that $1 \pm \phi_d(\omega') \geq 0$ for all d and all ω'. The assertion of part (i), applied with $\tilde{t}$ replacing t shows that

$$\lambda^{\omega}\phi_{a\Delta b}(\omega) \exp(H(\omega, \tilde{t}\,) \geq 0 \qquad \text{for all } \omega'.$$

□ Assertion (ii) follows.

1. Notes

The inequalities are originally due to Griffiths (1967). The proof comes from Ginibre (1969) via the exposition of Liggett (1985, Section IV.1).

References

Ginibre, J. (1969), ‘Simple proof and generalizations of Griffiths’ second inequality’, *Physical Review Letters* **23**, 828–830.

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