THE TWO-DIMENSIONAL ISING MODEL

The model has $S = \mathbb{Z}^2$ as its set of sites, with $\mathbb{F} = \{\{i, j\} \in S \times S : |i_1 - j_1| + |i_2 - j_2| = 1\}$. For each i, the dominating λ^i is counting measure on $\mathcal{X}_i = \{-1, +1\}$.

For each $A \in \mathbb{S}$ define $\phi_A(\omega) = \prod_{i \in A} \omega_i$. The set of Gibbs measures $\mathbb{G}(\beta)$ is defined by the functions

$$\psi_a(\omega) = \exp(\beta\phi_a(\omega_a)) \qquad \text{for } a \in \mathbb{F}$$

As the positive parameter β changes, the form of $\mathbb{G}(\beta)$ changes dramatically. In this handout I will explain some of the facts established in the literature.

For what follows, I will mostly choose $A(n)$ to be the 'square'

$$\{(i_1, i_2) \in S : \max(|i_1|, |i_2|) \le n\},$$

writing, as usual, $S(n)$ for $S\backslash A(n)$ and $N(n)$ for $\mathcal{N}(A(n))$.

The Gibbs measures in $\mathbb{G}(\beta)$ are defined to have the discrete conditional distributions $Q_{S\backslash A,\beta}(\cdot \mid \omega_{S\backslash A})$ defined by

$$Q_{S\backslash A,\beta}(f(\omega_A, \omega_{S\backslash A}) \mid \omega_{S\backslash A}) = \frac{1}{Z_{A,\beta}(\omega_{\mathcal{N}(A)})}\lambda^A f(\omega_A, \omega_{S\backslash A}) \exp\left(\beta \sum_{a \in \partial A} \phi_a(\omega)\right)$$

where $Z_{A,\beta}(\omega_{\mathcal{N}(A)})$ is a normalizing constant. If f depends only on ω_A then only the components of $\omega_{S\backslash A}$ for sites in $\mathcal{N}(A)$ are needed to define the integral. For that case, I will abuse notation by writing $Q_{S\backslash A}(f(\omega_A) \mid \omega_{\mathcal{N}(A)})$. In particular, the measure $Q_{S\backslash A,\beta}(\cdot \mid \omega_{\mathcal{N}(A)})$ puts mass

$$\frac{\exp\left(\beta \sum_{a \in \partial A} \phi_a(\omega)\right)}{Z_{A,\beta}(\omega_{\mathcal{N}(A)})} \qquad \text{at } \omega_A.$$

[§ising] 1. Things I can prove

- A real function on $\Omega := \prod_{i \in S} \mathcal{X}_i$ is said to be increasing if it is increasing in each argument.

(i) There exist two extremal measures, $\mathbb{P}_{\beta,+}$, $\mathbb{P}_{\beta,-}$, in $\mathbb{G}(\beta)$ for which

$$\mathbb{P}_{\beta,-} f \le \mu f \le \mathbb{P}_{\beta,+} f \qquad \text{for all } \mu \in \mathbb{G}(\beta), \text{ all increasing } f \in \mathcal{L}$$

[In fact, these are the only extremal Gibbs measures but I can't prove that. See Aizenman (1980).]

(ii) Both $\mathbb{P}_{\beta,+}$ and $\mathbb{P}_{\beta,-}$ are translation invariant.

(iii) $\mathbb{P}_{\beta,-} = \mathbb{P}_{\beta,+}$ if and only if $\mathbb{P}_{\beta,+}\omega_0 = 0$.

(iv) There exists some critical value $\beta_c \in (0, \infty)$ such that

$$\begin{cases} \mathbb{P}_{\beta,-} = \mathbb{P}_{\beta,+} & \text{for } 0 < \beta < \beta_c \\ \mathbb{P}_{\beta,-} \ne \mathbb{P}_{\beta,+} & \text{for } \beta > \beta_c \end{cases}$$

[If I understood Aizenman's proof I could also show that $\mathbb{G}(\beta)$ is a one-parameter family, $\{\theta\mathbb{P}_{\beta,-} + (1-\theta)\mathbb{P}_{\beta,+} : 0 \le \theta \le 1\}$ for $\beta > \beta_c$.]

[§dobrushin] 2. Uniqueness for small β

Use the Dobrushin uniqueness condition to show that there is only one Gibbs measure if β is close enough to zero.

[§extremal] 3. Two extremal Gibbs measures

Each μ in $\mathbb{G}(\beta)$ can be generated as a cluster point of a sequence of functionals $\mu_n = \nu_n Q_{S(n),\beta}(\cdot \mid \omega_{S(n)})$ on $\mathcal{L}$, for probability measures ν_n on $\mathcal{F}_{S(n)}$.

Remark. In one sense the assertion about cluster points is trivially true: by definition of a Gibbs measure, μ_n equals μ if we take ν_n to be the restriction of μ to $\mathcal{F}_{S(n)}$.

For each fixed f in $\mathcal{L}$, when the support of f is a subset of $A(n)$ the value $\mu_n f$ is completely determined by the restriction of ν_n to $\mathcal{F}_{N(n)}$. That is, we have only to specify the distribution over the 'boundary values' $\omega_{N(n)}$ to determine $\mu_n f$ for such an f.

The measures $\mathbb{P}_{\beta,-}$ and $\mathbb{P}_{\beta,+}$ are determined by the extreme boundary conditions: $\oplus_n$, which denotes the configuration of $\omega_{N(n)}$ with $\omega_i = +1$ for all $i \in N(n)$; and $\ominus_n$, which denotes the configuration of $\omega_{N(n)}$ with $\omega_i = -1$ for all $i \in N(n)$.

Use the Peierls argument to prove existence of the two distinct Gibbs measures when β is large enough: show that $\mathbb{P}_{\beta,-}\{\omega_0 = +1\} \le 1/3$ and $\mathbb{P}_{\beta,+}\{\omega_0 = +1\} \ge 2/3$ is β is large enough.

At the moment the notation is misleadingly specific because I have not yet shown that the cluster points are unique. The next Lemma will help me to establish convergence by means of a monotonicity property of the Q's.

fixedA <1> **Lemma.** *For a fixed finite subset A of $\mathbb{Z}^2$, let b and b^* be two possible boundary conditions for which $b \le b^*$, that is, $b_j \le b_j^*$ for all $j \in \mathcal{N}(A)$. Let f be an increasing function on Ω_A. Then*

$$Q_{S\backslash A,\beta}\left(f(\omega_A) \mid b\right) \le Q_{S\backslash A,\beta}\left(f(\omega_A) \mid b^*\right)$$

Compare with Liggett 1985, page 188.

Proof. For simplicity of notation, if $x \in \Omega_A$ define

$$p(x) = Q_{S\backslash A,\beta}\left(\omega_A = x \mid b\right) \qquad \text{and} \qquad p^*(x) = Q_{S\backslash A,\beta}\left(\omega_A = x \mid b^*\right).$$

see the handout FKG.pdf

By Holley's inequality, it is enough if we show that

$$p(x)p^*(y) \le p(x \wedge y)p^*(x \vee y) \qquad \text{for all } x, y \in \Omega_A.$$

Equivalently, we need to show

phi4 <2>
$$\sum_{a \in \partial A} \left(\phi_a(x \vee y, b^*) - \phi_a(y, b^*) + \phi_a(x \wedge y, b) - \phi_a(x, b)\right) \ge 0$$

In fact, it is not hard to show that each summand is nonnegative.

Consider first the case where $a = \{i, j\}$ with both i and j in A. Define

$$u_i = (x_i - y_i)^+ \qquad \text{and} \qquad u_j = (x_j - y_j)^+$$
$$v_i = x_i \wedge y_i \qquad \text{and} \qquad v_j = x_j \wedge y_j$$

Use the identities

$$s \vee t = t + (s-t)^+ \qquad \text{and} \qquad s = (s-t)^+ + s \wedge t \qquad \text{for all } s, t \in \mathbb{R}$$

to rewrite the ath summand in <2> as

$$\begin{aligned} &(y_i + u_i)\left(y_j + u_j\right) - y_i y_j + v_i v_j - (u_i + v_i)\left(u_j + v_j\right) \\ &= u_i y_j + u_j y_i - v_i u_j - v_j u_i \\ &= u_i(y_j - v_j) + u_j(y_i - v_i) \\ &\ge 0 \qquad \text{because } t - s \wedge t \ge 0 \text{ for all } s, t. \end{aligned}$$

For the case where $a = \{i, j\}$ with $i \in A$ and $j \in \mathcal{N}(A)$, the ath summand of <2> becomes

$$\begin{aligned}&(x_i \vee y_i)b_j^* - y_i b_j^* + (x_i \wedge y_i)b_j - x_i b_j\\ =&(y_i + u_i)b_j^* - y_i b_j^* + v_i b_j - (u_i + v_i)b_j\\ =&\ u_i(b_j^* - b_j) \geq 0\end{aligned}$$

□

Consider a fixed increasing function f in $\mathcal{L}_A$. When n is large enough that $A \subseteq A(n)$, the Lemma shows that for all $\omega_{N(n)}$,

$$Q_{S(n),\beta}\left(f \mid \ominus_n\right) \leq Q_{S(n),\beta}\left(f \mid \omega_{N(n)}\right) \leq Q_{S\setminus A,\beta}\left(f \mid \oplus_n\right)$$

If $\mu \in \mathbb{G}(\beta)$ we have $\mu f = \mu Q_{S(n),\beta}\left(f \mid \omega_{N(n)}\right)$. It follows, for all n large enough, that

$$Q_{S(n),\beta}\left(f \mid \ominus_n\right) \leq \mu f \leq Q_{S(n),\beta}\left(f \mid \oplus_n\right).$$

The consistency condition for the Q's gives an even better inequality:

$$\begin{aligned}Q_{S(n+1)}(f \mid \oplus_{n+1}) &= Q_{S(n+1)}\left(Q_{S(n)}(f \mid \omega_{N(n)}) \mid \oplus_{n+1}\right)\\ &\leq Q_{S(n)}(f \mid \oplus_n)\end{aligned}$$

Thus

downQ <3> $$Q_{S(n)}(f \mid \oplus_n) \downarrow \mathbb{P}_{\beta,+}f \qquad \text{for each increasing } f \text{ in } \mathcal{L}.$$

Similarly,

upQ <4> $$Q_{S(n)}(f \mid \ominus_n) \uparrow \mathbb{P}_{\beta,-}f \qquad \text{for each increasing } f \text{ in } \mathcal{L}.$$

In the limit we have

sandwich <5> $$\mathbb{P}_{\beta,-}f \leq \mu f \leq \mathbb{P}_{\beta,+}f \qquad \text{for each increasing } f \text{ in } \mathcal{L}.$$

span <6> **Lemma.** *The increasing functions in $\mathcal{L}_A$ span $\mathcal{L}_A$.*

cf. Kindermann & Snell (1980, page 129)

Proof. Note that

$$\omega_i\omega_j = (1+\omega_i)(1+\omega_j) - 1 - \omega_i - \omega_j,$$

a representation of $\omega_i\omega_j$ as a linear combination of increasing functions. Similarly,

$$\begin{aligned}\omega_i\omega_j\omega_k = &(1+\omega_i)(1+\omega_j)(1+\omega_k)\\ &- 1 - \omega_i - \omega_j - \omega_k - \omega_i\omega_j - \omega_j\omega_k - \omega_k\omega_i.\end{aligned}$$

And so on.

Thus $\Phi := \{\phi_D : D \subseteq A\}$ is contained in the linear span. Finish the argument by noting that Φ is an orthonormal basis for $\mathcal{L}_A$ under the uniform distribution on Ω_A. □

squish <7> **Corollary.** *The measures $\mathbb{P}_{\beta,-}$ and $\mathbb{P}_{\beta,+}$ are the same if and only if $\mathbb{P}_{\beta,-}f = \mathbb{P}_{\beta,+}f$ for each increasing f in $\mathcal{L}$.*

From <4>, <3>, and Lemma <6>, we get

$$Q_{S(n)}(f \mid \oplus_n) \to \mathbb{P}_{\beta,+}f \qquad \text{for each } f \in \mathcal{L}$$

That is, $\mathbb{P}_{\beta,+}$ is not just a cluster points of the $Q_{S(n)}(\cdot \mid \oplus_n)$ functionals, it is actually a limit. A similar assertion holds for $\mathbb{P}_{\beta,-}$.

The inequalities in <5> establish the extremality of both $\mathbb{P}_{\beta,-}$ and $\mathbb{P}_{\beta,+}$ in $\mathbb{G}(\beta)$. For example, suppose we have $\mathbb{P}_{\beta,-} = \theta\mu_1 + (1-\theta)\mu_2$ with $0 < \theta < 1$ and both μ_i in $\mathbb{G}(\beta)$. Then <5>, applied to each μ_i, forces $\mathbb{P}_{\beta,-}f = \mu_i f$ for all increasing f in $\mathcal{L}$. That is, $\mathbb{P}_{\beta,-}$ is extremal.

Inequalities <5> would also show that both extremal measures are translation invariant. Indeed, we could repeat the construction for the measures with $A(n)$ replaced by a square centered at another site, say s. We would then get another pair of extremal measures for which, for all $\mu \in \mathbb{G}(\beta)$,

sandwich.s <8>
$$\tilde{\mathbb{P}}_{\beta,-} f \le \mu f \le \tilde{\mathbb{P}}_{\beta,+} f \qquad \text{for each increasing } f \text{ in } \mathcal{L}.$$

In fact $\tilde{\mathbb{P}}_{\beta,+}$ would be the image of $\mathbb{P}_{\beta,+}$ under the translation map on S that takes 0 to s. An appeal to <5> with μ equal to $\tilde{\mathbb{P}}_{\beta,+}$, followed by an appeal to <8> with μ equal to $\mathbb{P}_{\beta,+}$, would then give $\mathbb{P}_{\beta,+} = \tilde{\mathbb{P}}_{\beta,+}$. And so on.

[§monotonicity]

4. Monotonicity in β

There is a unique Gibbs measure in $\mathbb{G}(\beta)$ if and only if $\mathbb{P}_{\beta,-} = \mathbb{P}_{\beta,+}$. We know that

$$\mathbb{P}_{\beta,-} f \le \mathbb{P}_{\beta,+} f \qquad \text{for each increasing } f \text{ in } \mathcal{L}.$$

The next Lemma will show that the two measures are equal if and only if $\mathbb{P}_{\beta,-}\omega_i = \mathbb{P}_{\beta,+}\omega_i$ for each $i \in S$. By translation invariance, it is enough to check this equality for a single site, such as the origin. That is, $\mathbb{P}_{\beta,-} = \mathbb{P}_{\beta,+}$ if and only if $\mathbb{P}_{\beta,-}\omega_0 = \mathbb{P}_{\beta,+}\omega_0$. By symmetry,

$$\mathbb{P}_{\beta,+}\{\omega_0 = +1\} = \mathbb{P}_{\beta,-}\{\omega_0 = -1\} = 1 - \mathbb{P}_{\beta,-}\{\omega_0 = +1\}$$

The two extremal measures are the same precisely when $\mathbb{P}_{\beta,+}\{\omega_0 = +1\} = 1/2$, that is, when $\mathbb{P}_{\beta,+}\omega_0 = 0$.

Giffiths.pdf

Griffiths's inequality shows that $Q_{S(n),\beta}(\omega_0 \mid \oplus_n)$ is an increasing function of β for each n. In the limit, $\mathbb{P}_{\beta,+}\omega_0$ must also be an increasing function of β. The set $\{\beta > 0 : \mathbb{P}_{\beta,+}\omega_0 > 0\}$ must be an interval of the form $[\beta_c, \infty)$ or (β_c, ∞), for some positive β_c.

stoch.order <9> **Lemma.** *Suppose P and Q are probability measures on Ω_A for which $Pf \le Qf$, for each increasing function f on Ω_A. Then $P = Q$ if and only if $P\omega_i = Q\omega_i$ for each $i \in A$.*

Proof. Necessity is trivial.

Suppose D is a nonempty subset of A. For convenience suppose the sites in D are $\omega_1, \dots, \omega_k$. Invoke the inequalities for $f(\omega_A)$ equal to $\omega_1 \vee \omega_2$ and then equal to $\omega_1 \wedge \omega_2$ to get

$$P\omega_1 + P\omega_2 = P(\omega_1 \wedge \omega_2) + P(\omega_1 \vee \omega_2) \le Q(\omega_1 \wedge \omega_2) + Q(\omega_1 \vee \omega_2) = Q\omega_1 + Q\omega_2$$

The equality $P\omega_1 + P\omega_2 = Q\omega_1 + Q\omega_2$ then forces $P(\omega_1 \wedge \omega_2) = Q(\omega_1 \wedge \omega_2)$. Similarly

$$\begin{aligned} P(\omega_1 \wedge \omega_2) + P\omega_3 &= P(\omega_1 \wedge \omega_2 \wedge \omega_3) + P\big((\omega_1 \wedge \omega_2) \vee \omega_3\big) \\ &\le Q(\omega_1 \wedge \omega_2 \wedge \omega_3) + Q\big((\omega_1 \wedge \omega_2) \vee \omega_3\big) \\ &= Q(\omega_1 \wedge \omega_2) + Q\omega_3, \end{aligned}$$

which forces $P(\omega_1 \wedge \omega_2 \wedge \omega_3) = Q(\omega_1 \wedge \omega_2 \wedge \omega_3)$. And so on.

Thus we have

$$P\gamma_D = \tfrac{1}{2}P(1 + \min_{i \in D} \omega_i) = \tfrac{1}{2}Q(1 + \min_{i \in D} \omega_i) = Q\gamma_D$$

for each $D \subseteq A$. An appeal to a π–λ theorem then gives $P = Q$, as measures on Ω_A. □

[§notes] ## 5. Notes

See Kindermann & Snell (1980, Appendix 1) for calculations with the Ising model. See Georgii (1988, Chapter 6) for a detailed, rigorous analysis of the Ising model. See Lebowitz & Martin-Löf (1972) or Ruelle (1972) for a characterization of the β_c value at which phase transition begins.

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