Thus

$$p_{A^c}(\omega_{A^c}) = \lambda^A p(\omega_A, \omega_{A^c}) = \frac{Z_{\mathcal{N}(A)}}{Z} \prod\nolimits_{a \in \mathbb{F}\backslash \partial A} \psi_a(\omega_a)$$
$$\text{where } Z_{\mathcal{N}(A)} = Z_{\mathcal{N}(A)}(\omega_{\mathcal{N}(A)}) := \lambda^A \prod\nolimits_{a \in \partial A} \Psi_a(\omega_a)$$

and

$$\begin{aligned} p_{A|A^c}(\omega_A \mid \omega_{A^c}) &= \tilde{p}_{\mathcal{N}(A)}(\omega_A \mid \omega_{\mathcal{N}(A)}) \\ &:= \frac{\{Z_{\mathcal{N}(A)} \neq 0\}}{Z_{\mathcal{N}(A)}} \prod\nolimits_{a \in \partial A} \Psi_a(\omega_a) \qquad \text{a.e. } [\lambda^A \otimes \mathbb{P}^{A^c}]. \end{aligned}$$

Notice that $\tilde{p}_{\mathcal{N}(A)}$ depends on ω only through the coordinates $\omega_{A \cup \mathcal{N}(A)}$. In fact, it defines a new Markov kernel, a family of measures $Q_{\mathcal{N}(A)}(\cdot \mid \omega_{\mathcal{N}(A)})$ on $\mathcal{B}_A$: for each $\omega_{\mathcal{N}(A)}$,

$$\frac{dQ_{\mathcal{N}(A)}}{d\lambda^A} = \tilde{p}_{\mathcal{N}(A)}(\omega_A \mid \omega_{\mathcal{N}(A)})$$

Notice that $Q_{\mathcal{N}(A)}(\cdot \mid \omega_{\mathcal{N}(A)})$ is a probability measure at each $\omega_{\mathcal{N}(A)}$ for which $Z_{\mathcal{N}(A)}(\omega_{\mathcal{N}(A)}) \neq 0$.

For $D \subseteq A$, write $Q^D_{\mathcal{N}(A)}$ for the marginal distribution of ω_D under $Q_{\mathcal{N}(A)}$. The new Markov kernels fit together in an elegant way.

<14> **Lemma.** *Suppose $A \subset B$. Then, at least for nonnegative, $\mathcal{B}_A$-measurable f,*

$$Q^{B\mathcal{N}(A)}_{\mathcal{N}(B)} Q^A_{\mathcal{N}(A)} f(\omega_A) = Q^A_{\mathcal{N}(B)} f(\omega_A)$$

REMARK. The $Q^{B\mathcal{N}(A)}_{\mathcal{N}(B)}$ notation denotes an integral with respect to the $Q_{\mathcal{N}(B)}$ distribution, for a fixed $\omega_{\mathcal{N}(B)}$, over the ω_i coordinates for $i \in B \cap \mathcal{N}(A)$. The coordinates in $\mathcal{N}(A)\backslash B$ are held fixed.

Proof. For later applications, I think it will be enough to have the result when $B \supseteq A \cup \mathcal{N}(A)$. I will prove only that case. The proof for case where $\mathcal{N}(A)\backslash B \neq \emptyset$ is similar but involves slightly more notation.

Let me simplify notation by defining $N = \mathcal{N}(A)$ and $M = \mathcal{N}(B)$ and $D = B\backslash \big(A \cup N\big)$, so that B is a union of disjoint sets $A \cup N \cup D$. We need to show that

<15> $$Q^N_M Q^A_N f(\omega_A) = Q_M f.$$

Also note the dependence of the densities on particular blocks of coordinates by writing

$$\prod\nolimits_{a \in \partial A} \Psi_a(\omega_a) = G(\omega_A, \omega_N)$$
$$\prod\nolimits_{a \in \partial B \backslash \partial A} \Psi_a(\omega_a) = h(\omega_D, \omega_N, \omega_M)$$

so that

$$\begin{aligned} Z_N(\omega_N) &= \lambda^A G(\omega_A, \omega_N) \\ Z_M(\omega_M) &= \lambda^D \otimes \lambda^N \otimes \lambda^A G(\omega_A, \omega_N) h(\omega_D, \omega_N, \omega_M) \\ &= \lambda^N Z_N(\omega_N) H(\omega_N, \omega_M) \end{aligned}$$

where

$$H(\omega_N, \omega_M) := \lambda^D h(\omega_D, \omega_N, \omega_M)$$

The asserted equality <15> holds trivially (both sides are zero) when $Z_M(\omega_M)$ is zero. For the rest of the proof consider a fixed ω_M at which $Z_M = Z_M(\omega_M) \neq 0$. From the definition of Q_M on $\mathcal{B}_B$,

$$\frac{dQ_M}{d\big(\lambda^D \otimes \lambda^N \otimes \lambda^A\big)} = \frac{1}{Z_M} G(\omega_A, \omega_N) h(\omega_D, \omega_N, \omega_M)$$

The density of the marginal distribution of ω_N under Q_M is obtained by integrating out with respect to $\lambda^D \otimes \lambda^A$:

$$\frac{dQ_M^N}{d\lambda^N} = \frac{1}{Z_M}\lambda^D\lambda^A G(\omega_A,\omega_N)H(\omega_D,\omega_N,\omega_M) = \frac{Z_N(\omega_N)}{Z_M}H(\omega_N,\omega_M)$$

Thus the left-hand side of <15> equals

$$\begin{aligned}\lambda^N \frac{Z_N(\omega_N)}{Z_M}H(\omega_N,\omega_M)&\frac{\{Z_N(\omega_N)\neq 0\}}{Z_N(\omega_N)}\lambda^A G(\omega_A,\omega_N)f(\omega_A)\\ &= \lambda^N\lambda^A\frac{\{Z_N(\omega_N)\neq 0\}}{Z_M}H(\omega_N,\omega_M)G(\omega_A,\omega_N)f(\omega_A)\end{aligned}$$

On the set $\{Z_N(\omega_N)=0\}$ we have $G(\omega_A,\omega_N)=0$ a.e. $[\lambda^A]$. The last iterated integral is, therefore, unchanged if we omit the indicator function, leaving

$$\begin{aligned}&\frac{1}{Z_M}\lambda^N\lambda^A H(\omega_N,\omega_M)G(\omega_A,\omega_N)f(\omega_A)\\ &\qquad = \frac{1}{Z_M}\lambda^N\lambda^A\lambda^D h(\omega_D,\omega_N,\omega_M)G(\omega_A,\omega_N)f(\omega_A) = Q_M f,\end{aligned}$$

□ as asserted by <15>.

6. Conditional distributions for countable sets of sites

When S is countably infinite it is not always possible to define a joint density for ω_S by taking an infinite product.

<16> **Example.** Suppose $\mathcal{X}_i = \{0,1\}$ and λ^i is the uniform distribution on $\mathcal{X}_i$, for every $i \in S = \mathbb{N}$. Suppose $\mathbb{F} = \{\{i,i+1\} : i \in S\}$ and

better example needed

$$\Psi_{\{i,i+1\}} = 2\{\omega_i = \omega_{i+1}\} + \{\omega_i \neq \omega_{i+1}\}.$$

The product measure $\lambda^S = \otimes_{i\in S}\lambda^i$ is well defined. We might hope to construct $\mathbb{P}$ by defining

$$\frac{d\mathbb{P}}{d\lambda^S}(\omega_S) = \frac{1}{Z}\prod\nolimits_{i\in S}\Psi_{\{i,i+1\}}(\omega_i,\omega_{i+1})$$
$$\text{where } Z = \lambda^S\prod\nolimits_{i\in S}\Psi_{\{i,i+1\}}(\omega_i,\omega_{i+1}).$$

Unfortunately,

$$Z \geq \prod\nolimits_{j\in\mathbb{N}}\lambda\otimes\lambda\Psi_{\{2j,2j+1\}}(\omega_{2j},\omega_{2j+1}) = \prod\nolimits_{j\in\mathbb{N}}\left(\tfrac{1}{2}\times 2 + \tfrac{1}{2}\times 1\right) = \infty$$

□ We would end up with ∞/∞.

An alternative method is to construct $\mathbb{P}$ as some sort of limit of distributions on larger and larger finite chunks of S. The use of $Q_{\mathcal{N}}(A)$ conditional distributions will help us to avoid an inconsistent assignment of conditional distributions.

<17> **Example.** Suppose $S = \{1,2\}$ with $\mathcal{X}_i = \{0,1\}$ and λ^i equal to counting measure, for each i. Does there exist a probability density $p(\omega_1,\omega_2)$ on $\mathcal{X}_1\times\mathcal{X}_2$ with conditionaal densities

$$\begin{aligned}p_{1|2}(\omega_1 \mid \omega_2) &= \alpha\{\omega_1=\omega_2\} + \overline{\alpha}\{\omega_1\neq\omega_2\} \qquad \text{where } \overline{\alpha} = 1-\alpha\\ p_{2|1}(\omega_2 \mid \omega_1) &= \beta\{\omega_1=\omega_2\} + \overline{\beta}\{\omega_1\neq\omega_2\} \qquad \text{where } \overline{\beta} = 1-\beta\end{aligned}$$

where $\alpha \neq \beta$? If there were such a p we would have

$$p(0,0) + p(1,1) = p_1(0)p_{2|1}(0 \mid 0) + p_1(1)p_{2|1}(1 \mid 1) = \beta$$

A similar argument with the roles of the two coordinates interchanged would give $p(0,0) + p(1,1) = \alpha$, a contradiction. □

Everything from here onwards is under revision.

<18> **Definition.** *Write $\mathcal{S}$ for the collection of all finite, nonempty subsets of S. Write $\mathcal{L}$ for the set of all bounded, $\mathcal{B}$-measurable, real functions on Ω that depend on ω only through a finite set of coordinates $S(f) \subset S$. For each $A \in \mathcal{S}$, write $\mathcal{L}_A$ for $\{f \in \mathcal{L} : S(f) \subseteq A\}$. Write m_f for $\sup_\omega |f(\omega)|$.*

cf. Georgii (1988, page 31)

7. Notes

See Griffeath (1976) for the main idea (the so-called Möbius inversion, as presented in Lemma <12>) behind the proof of the Hammersley-Clifford result.

Georgii (1988, page 16) called a family of Markov kernels that satisfies a condition like the one asserted by Lemma <14> a ***specification***.

References

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