Chapter 2

A modicum of measure theory

2 February 2004: Modification of Section 2.11.

*1. Generating classes of functions

Theorem <DYNKIN.THM> is often used as the starting point for proving facts about measurable functions. One first invokes the Theorem to establish a property for sets in a sigma-field, then one extends by taking limits of simple functions to $\mathcal{M}^+$ and beyond, using Monotone Convergence and linearity arguments. Sometimes it is simpler to invoke an analog of the λ-system property for classes of functions.

<1> **Definition.** *Let $\mathcal{H}$ be a set of bounded, real-valued functions on a set $\mathcal{X}$. Call $\mathcal{H}$ a* ***λ-space*** *if:*

(i) $\mathcal{H}$ is a vector space

(i) each constant function belongs to $\mathcal{H}$;

(ii) if $\{h_n\}$ is an increasing sequence of functions in $\mathcal{H}$ whose pointwise limit h is bounded then $h \in \mathcal{H}$.

The sigma-field properties of λ-spaces are slightly harder to establish than their λ-system analogs, but the reward of more streamlined proofs will make the extra, one-time effort worthwhile. First we need an analog of the fact that a λ-system that is stable under finite intersections is also a sigma-field.

Remember that $\sigma(\mathcal{H})$ is the smallest σ-field on $\mathcal{X}$ for which each h in $\mathcal{H}$ is $\sigma(\mathcal{H})\backslash\mathcal{B}(\mathbb{R})$-measurable. It is the σ-field generated by the collection of sets $\{h \in B\}$ with $h \in \mathcal{H}$ and $B \in \mathcal{B}(\mathbb{R})$. It is also generated by

$$\mathcal{E}_{\mathcal{H}} := \{\{h < c\} : h \in \mathcal{H},\ c \in \mathbb{R}\}.$$

<2> **Lemma.** *If a λ-space $\mathcal{H}$ is stable under the formation of pointwise products of pairs of functions then it consists of all bounded, $\sigma(\mathcal{H})$-measurable functions.*

Proof. By definition, every function in $\mathcal{H}$ is $\sigma(\mathcal{H})$-measurable. The proof that every bounded, $\sigma(\mathcal{H})$-measurable function belongs to $\mathcal{H}$ will follow from the following four facts:

(a) $\mathcal{H}$ is stable under uniform limits

(b) if h_1 and h_2 are in $\mathcal{H}$ then so are $h_1 \vee h_2$ and $h_1 \wedge h_2$

(c) the collection of sets $\mathcal{A}_0 := \{A \in \mathcal{A} : A \in \mathcal{H}\}$ is a σ-field

(d) $\mathcal{E}_{\mathcal{H}} \subseteq \mathcal{A}_0$ and hence $\sigma(\mathcal{H}) = \sigma(\mathcal{E}_{\mathcal{H}}) \subseteq \mathcal{A}_0$

For suppose g is a bounded, $\sigma(\mathcal{H})$-measurable function. With no loss of generality (or by means of some linear rescaling) we may assume that $0 \le g \le 1$. For each real c, the (indicator function of the) $\sigma(\mathcal{H})$-measurable set $\{g \ge c\}$ belongs to $\mathcal{H}$, by virtue of (d) and (c). The vector space property of $\mathcal{H}$ ensures that the simple function $g_n := 2^{-n}\sum_{i=1}^{2^n}\{g \ge i/2^n\}$ also belongs to $\mathcal{H}$. Stability of $\mathcal{H}$ under uniform limits then implies that $g \in \mathcal{H}$.

Proof of (a). Suppose $h_n \to h$ uniformly, with $h_n \in \mathcal{H}$. Write δ_n for 2^{-n}. With no loss of generality we may suppose $h_n + \delta_n \ge h \ge h_n - \delta_n$ for all n. Notice that

$$h_n + 3\delta_n = h_n + \delta_n + \delta_{n-1} \ge h + \delta_{n-1} \ge h_{n-1}.$$

the functions $g_n := h_n + 3(\delta_1 + \ldots + \delta_n)$ all belong to $\mathcal{H}$, and $g_n \uparrow h + 3$. It follows that $h + 3 \in \mathcal{H}$, and hence, $h \in \mathcal{H}$.

Proof of (b). It is enough if we show that $h^+ \in \mathcal{H}$ for each h in $\mathcal{H}$, because $h_1 \vee h_2 = h_1 + (h_2 - h_1)^+$ and $-\big(h_1 \wedge h_2\big) = (-h_1) \vee (-h_2)$. Suppose $c \le h \le d$, for constants c and d. First note that, for every polynomial $p(y) = a_0 + a_1 y \ldots + a_m y^m$, we have

$$p(h) = a_0 + a_1 h + \ldots + a_m h^m \in \mathcal{H},$$

because the constant function a_0 and each of the powers h^k belong to the vector space $\mathcal{H}$. By a minor extension of the Weierstrass approximation result from Problem [WEIERSTRASS], the continuous function $y \mapsto y^+$ can be uniformly approximated by a polynomial on the interval $[c, d]$. That is, there exists a sequence of polynomials p_n such that $\sup_{c \le y \le d} |p_n(y) - y^+| \to 0$ as $n \to \infty$. In particular, h^+ is a uniform limit of $p_n(h)$, so that $h^+ \in \mathcal{H}$ by virtue of (a).

Proof of (c). The fact that $1 \in \mathcal{H}$ and the stability of $\mathcal{H}$ under monotone limits, differences, and finite products implies that $\mathcal{A}_0$ is a λ-system of sets that is stable under finite intersections, that is, $\mathcal{A}_0$ is a σ-field.

Proof of (d). Suppose $h \in \mathcal{H}$ and $c \in \mathbb{R}$. By (b), the function

$$h_0 := \big(1 + h - c\big)^+ \wedge 1$$

belongs to $\mathcal{H}$. Notice that $0 \le h_0 \le 1$ and $\{h_0 = 1\} = \{h \ge c\}$. As a monotone increasing limit of functions $1 - h_0^n$ from $\mathcal{H}$, the (indicator function of the) set $\{h < c\}$ also belongs to $\mathcal{H}$. □

<3> **Theorem.** *Let $\mathcal{G}$ be a set of functions from a λ-space $\mathcal{H}$. If $\mathcal{G}$ is stable under the formation of pointwise products of pairs of functions then $\mathcal{H}$ contains all bounded, $\sigma(\mathcal{G})$-measurable functions.*

Proof. Let $\mathcal{H}_0$ be the smallest λ-space containing $\mathcal{G}$. By Lemma <2>, it is enough to show that $\mathcal{H}_0$ is stable under pairwise products.

Argue as in Theorem <DYNKIN.THM> for λ-systems of sets. An almost routine calculation shows that $\mathcal{H}_1 := \{h \in \mathcal{H}_0 : hg \in \mathcal{H}_0 \text{ for all } g \text{ in } \mathcal{G}\}$ is a λ-space containing $\mathcal{G}$. The only subtlety lies in showing that $\mathcal{H}_1$ is stable under monotone increasing limits. If $h_n \in \mathcal{H}_1$ and $h_n \uparrow h$ and $g \geq 0$, then $gh_n \uparrow gh$. At points where g is strictly negative, the sequence gh_n would not be increasing. However, we can find a constant C large enough that $g + C \geq 0$ everywhere, and hence gh belongs to $\mathcal{H}_0$ as a monotone inceasing limit of $\mathcal{H}_0$ functions $h_n g + Ch_n - Ch$. It follows that $\mathcal{H}_1 = \mathcal{H}_0$. That is, $h_0 g \in \mathcal{H}_0$ for all $h_0 \in \mathcal{H}_0$ and $g \in \mathcal{G}$.

Similarly, $\mathcal{H}_2 := \{h \in \mathcal{H}_0 : h_0 h \in \mathcal{H}_0 \text{ for all } h_0 \text{ in } \mathcal{H}_0\}$ is a λ-space. By the result for $\mathcal{H}_1$ we have $\mathcal{H}_2 \supseteq \mathcal{G}$, and hence $\mathcal{H}_2 = \mathcal{H}_0$. That is, $\mathcal{H}_0$ is stable under products. □

<4> **Exercise.** Let μ be a finite measure on $\mathcal{B}(\mathbb{R}^k)$. Write $\mathbb{C}_0$ for the vector space of all continuous real functions on $\mathbb{R}^k$ with compact support. Suppose f belongs to $\mathcal{L}^1(\mu)$. Show that for each $\epsilon > 0$ there exists a g in $\mathbb{C}_0$ such that $\mu|f - g| < \epsilon$. That is, show that $\mathbb{C}_0$ is dense in $\mathcal{L}^1(\mu)$ under its $\mathcal{L}^1$ norm. □

SOLUTION: Define $\mathcal{H}$ as the collection of all bounded functions in $\mathcal{L}^1(\mu)$ that can be approximated arbitrarily closely (in $\mathcal{L}^1(\mu)$ norm) by functions from $\mathbb{C}_0$. Check that $\mathcal{H}$ is a λ-space. Trivially it contains $\mathbb{C}_0$. The sigma-field $\sigma(\mathbb{C}_0)$ coincides with the Borel sigma-field. Why? The class $\mathcal{H}$ consists of all bounded, nonnegative Borel measurable functions.

More detail needed?

See Problem [C0.DENSE2] for the extension of the approximation result to infinite measures. □