The theory for chains on general state spaces mirrors the theory for chains on countable state spaces when there is an accessible atom. Even if there is no such atom, it is possible to create one for a closely related chain when there exists a P-accessible set $A \in \mathcal{B}$ for which there exists a probability measure ν concentrated on A and a $\delta > 0$ for which

<1> $$P_x(\cdot) \ge \delta\nu(\cdot) \qquad \text{for all } x \in A.$$

Equivalently, with $\bar{\delta} = 1 - \delta$,

$$P_x(\cdot) = \delta\nu(\cdot) + \bar{\delta}R_x(\cdot) \qquad \text{for all } x \in A.$$

where $\{R_x : x \in \mathcal{X}\}$ is Markov kernel with $R_x = P_x$ for $x \in A^c$.

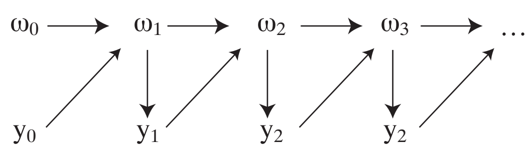
Define $s(x) = \delta\{x \in A\}$. Let γ denote the Bern(δ) distribution and γ_x denote the Bern$(s(x)) = \{x \in A\}\gamma$ distribution. For each $(x, y) \in \mathcal{X} \times \{0, 1\}$ define a probability $G_{x,y}$ on $\mathcal{B}$,

$$G_{x,y}(\cdot) = \{x \in A, y = 1\}\nu(\cdot) + \{x \in A, y = 0\}R_x(\cdot) + \{x \notin A\}P_x(\cdot).$$

Define a new Markov kernel $Q_{x,y}^{x',y'} = G_{x,y}^{x'}\gamma_{x'}^{y'}$. That is, to generate (x', y') from $Q_{x,y}$

$$\text{generate } x' \mid x, y \sim G_{x,y}$$
$$\text{generate } y' \mid x, y, x' \sim \gamma_{x'}$$

More formally, $Q_{x,y} f(x', y') = G_{x,y}^{x'}\gamma_{x'}^{y'} f(x', y')$.



The Q-chain has state space $\mathcal{Z} := \mathcal{X} \times \{0, 1\}$. The set $\alpha := A \times \{1\}$ is an atom: $Q_{x,y} = \nu \otimes \text{Bern}(\delta)$ for $(x, y) \in \alpha$. Also

$$\begin{aligned}
\gamma_x^y G_{x,y} &= s(x)G_{x,1} + \bar{s}(x)G_{x,0} \qquad \text{where } \bar{s}(x) := 1 - s(x) \\
&= \{x \in A\}\left(\delta\nu + \bar{\delta}R_x\right) + \{x \notin A\}P_x \\
&= P_x
\end{aligned}$$

This last fact implies that the Q-chain has a P-chain as its ω-marginal:

$$\begin{aligned}
&\mathbb{Q}_{x,y} f(\omega_0, \omega_1, \ldots, \omega_n) \\
&\quad = G_{x,y}^{\omega_1}\gamma_{\omega_1}^{y_1}G_{\omega_1,y_1}^{\omega_2} \ldots \gamma_{\omega_{n-1}}^{y_{n-1}}G_{\omega_{n-1},y_{n-1}}^{\omega_n}\gamma_{\omega_n}^{y_n} f(x, \omega_1, \ldots, \omega_n) \\
&\quad = G_{x,y}^{\omega_1}P_{\omega_1}^{\omega_2} \ldots P_{\omega_{n-1}}^{\omega_n} f(x, \omega_1, \ldots, \omega_n) \\
&\quad = \mathbb{P}_\mu f(x, \omega_1, \ldots, \omega_n) \qquad \text{where } \mu := G_{x,y}
\end{aligned}$$

Consequently,

$$\mathbb{Q}_{x,\gamma} f(\omega_0, \omega_1, \ldots, \omega_n) = \mathbb{P}_x f(\omega_0, \omega_1, \ldots, \omega_n)$$

If the set A is P-accessible from each $x \in \mathcal{X}$ then α is accessible from each (x, y) in $\mathcal{Z}_\alpha := \left(\mathcal{X} \times \{0\}\right) \cup \alpha$.

<2> **Theorem.** *Suppose μ is an invariant measure for the Q-chain. Then μ_0, the $\mathcal{X}$-marginal of μ, is an invariannt probability measure for the P-chain.*

Proof. Note that $Q_{x,y}A^c \times \{1\} = 0$ for all (x, y) and, for each $B \subseteq A$,

$$Q_{x,y}\left(B \times \{1\}\right) = \delta G_{x,y}B$$
$$Q_{x,y}\left(B \times \{0\}\right) = \bar{\delta} G_{x,y}B$$

It follows that

$$\mu\left(B \times \{1\}\right) = \mu^{x,y}Q_{x,y}\left(B \times \{1\}\right) = \delta\mu^{x,y}G_{x,y}B$$
$$\mu\left(B \times \{0\}\right) = \mu^{x,y}Q_{x,y}\left(B \times \{0\}\right) = \bar{\delta}\mu^{x,y}G_{x,y}B$$

Add to get $\mu_0 B = \mu^{x,y}G_{x,y}B$ and hence

$$\mu\left(B \times \{1\}\right) = \delta\mu_0 B \qquad \text{and} \qquad \mu\left(B \times \{0\}\right) = \bar{\delta}\mu_0 B \qquad \text{for } B \subseteq A.$$

Thus

$$\begin{aligned} \mu_0 f &= \mu^{x,y}Q_{x,y}^{x',y'} f(x') \\ &= \mu^{x,y}G_{x,y}^{x'} f(x') \\ &= \mu^{x,y}\left(\{x \in A, y = 1\}\nu f + \{x \in A, y = 0\}R_x f + \{x \notin A\}P_x f\right) \\ &= \mu_0^x\left(\{x \in A\}\delta\nu f + \{x \in A\}\bar{\delta}R_x f + \{x \notin A\}P_x f\right) \\ &= \mu_0^x P_x f \end{aligned}$$

□

1. Notes

The construction of the split chain is due to Nummelin (1978). The books of Nummelin (1984, Section 4.4) and Meyn & Tweedie (1993, Section 5.1) describe the construction in slightly different ways.

References

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