1. Notation

Start with a Markov kernel $P(x, \cdot)$ defined for sigma-field $\mathcal{B}$ on a state space $\mathcal{X}$. For $k \in \mathbb{N}$ define the ***k-step transition probabilities*** $P^{(k)}(x, \cdot)$ by $P^{(1)}(x, \cdot) = P(x, \cdot)$ and

$$P^{(k)}(x, \cdot) = P_x^y P^{(k-1)}(y, \cdot)$$

That is, $P^{(k)}(x, f) = \mathbb{P}_x f(X_k)$. For each k, the Markov kernel $P^{(k)} P(x, \cdot)$ gives the transition probabilities for a P-chain observed at times $0, k, 2k, \ldots$.

Define a new Markov kernel by

$$K(x, f) = \sum\nolimits_{n\in\mathbb{N}} 2^{-n} P^{(n)}(x, f) = \sum\nolimits_{n\in\mathbb{N}} 2^{-n} \mathbb{P}_x f(X_n)$$

Note that $\mathbb{P}_x\{\tau_A < \infty\} > 0$ if and only if $K(x, A) > 0$.

A set $\alpha \in \mathcal{B}$ is said to be an ***atom*** for the Markov kernel if $P(x, \cdot) = P(x', \cdot)$ for all $x, x' \in \alpha$. Write P_α for the common distribution P_x and $\mathbb{P}_\alpha$ for the common distribution $\mathbb{P}_x$ for all $x \in \alpha$. The atom is said to be ***accessible*** if $x \curvearrowright \alpha$ for all $x \in \mathcal{X}$, that is, if $\mathbb{P}_x\{\tau_\alpha < \infty\} > 0$ for all $x \in \mathcal{X}$, in which case the function $f_0(x) := K_x\alpha$ is everywhere strictly positive.

2. Subinvariant measures

A measure μ on $\mathcal{B}$ is said to be ***subinvariant*** for the Markov kernel P if $\mu^x P_x \le \mu$, that is, $\mu^x P_x^y f(y) \le \mu f$ for all $f \in \mathcal{M}^+(\mathcal{B})$.

<1> **Lemma.** *Suppose μ is subinvariant for P. Then*

(i) $\mu^x P^{(n)}(x, f) \le \mu f$ for each $f \in \mathcal{M}^+(\mathcal{B})$.

(ii) $\mu^x K_x f \le \mu f$ for each $f \in \mathcal{M}^+(\mathcal{B})$.

(iii) If α is an accessible atom then μ is sigma-finite.

Proof. For (i) argue inductively. For (ii) take weighted sums from (i). For (iii), note that accessibilty of α gives $K_x\alpha > 0$ for every $x \in \mathcal{X}$ and hence $1 \ge \lambda\alpha \ge \epsilon\mu\{x : K_x\alpha \ge \epsilon\}$ for each $\epsilon > 0$. □

<2> **Theorem.** *Suppose α is an accessible atom for the Markov kernel. Define a measure λ on $\mathcal{B}$ by*

$$\lambda f = \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha f(\omega_n)\{n \le \tau_\alpha\} \qquad \text{for each } f \in \mathcal{M}^+(\mathcal{B}).$$

(i) λ is a subinvariant, sigma-finite measure with $\lambda\alpha = \mathbb{P}_\alpha\{\tau_\alpha < \infty\} \le 1$ and $\lambda\mathcal{X} = \mathbb{P}_\alpha\tau_\alpha$.

(ii) If μ is a subinvariant measure then $\mu \ge (\nu\alpha)\lambda$.

(iii) λ is invariant if and only if α is recurrent, that is, $\lambda\alpha = \mathbb{P}_\alpha\{\tau_\alpha < \infty\} = 1$. In that case, λ is the unique subinvariant measure giving mass 1 to the set α.

(iv) If there exists a finite subinvariant measure μ with $\mu\alpha > 0$ then $c^{-1} := \mathbb{P}_\alpha\tau_\alpha$ is finite and $\lambda\alpha = 1$, the atom α is recurrent, and $c\lambda$ is an invariant probability measure.

Proof. Note that $Pf(\omega_n) = \mathbb{P}_{\omega_n} f(\omega_{n+1})$. Thus

$$\begin{aligned}
\lambda(Pf) &= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n \le \tau_\alpha\}\mathbb{P}_{\omega_n} f(\omega_{n+1}) \\
&= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n \le \tau_\alpha\} f(\omega_{n+1}) \qquad \text{by Markov property} \\
&= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n = \tau_\alpha\} f(\omega_{\tau+1}) + \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n+1 \le \tau_\alpha\} f(\omega_{n+1}) \\
&= \mathbb{P}_\alpha\{\tau_\alpha < \infty\} f(\omega_{\tau+1}) + \sum\nolimits_{n\ge 2} \mathbb{P}_\alpha\{n \le \tau_\alpha\} f(\omega_n).
\end{aligned}$$

By the strong Markov property, the first term equals $\mathbb{P}_\alpha\{\tau < \infty\} P_\alpha f$. The last sum is the same as the sum for λf except that the first term, $\mathbb{P}_\alpha f(\omega_1)\{\tau_\alpha \ge 1\} = P_\alpha f$, is missing. Add $\big(P_\alpha f\big)\mathbb{P}_\alpha\{\tau = \infty\}$ to both sides to get

<3>
$$\lambda(Pf) + \big(P_\alpha f\big)\mathbb{P}_\alpha\{\tau = \infty\} = \lambda f.$$

Clearly λ is subinvariant. Sigma-finiteness of λ follows from the fact that

$$\lambda\alpha = \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{\omega_n \in \alpha\}\{n \le \tau_\alpha\} = \mathbb{P}_\alpha\{\tau_\alpha < \infty\} \le 1$$

because $\omega_n \notin \alpha$ for $n < \tau_\alpha$. The measure λ has total mass

$$\lambda\mathcal{X} = \mathbb{P}_\alpha \sum\nolimits_{n\in\mathbb{N}}\{n \le \tau_\alpha\} = \mathbb{P}_\alpha\tau_\alpha.$$

To establish assertion (ii), suppose μ is a subinvariant measure. Write C for $\mu\alpha$. For a fixed $f_1 \in \mathcal{M}^+(\mathcal{B})$,

$$\begin{aligned}
\mu f_1 \ge \mu P f_1 &= \mu^x\{x \in \alpha\} P_x f_1 + \mu^x\{x \notin \alpha\} P_x f_1 \\
&\ge C P_\alpha f_1 + \mu P f_2 \qquad \text{where } f_2(x) := \{x \notin \alpha\} P_x f_1.
\end{aligned}$$

A similar argument gives

$$\mu P f_2 \ge C P_\alpha f_2 + \mu P f_3 \qquad \text{where } f_3(x) := \{x \notin \alpha\} P_x f_2.$$

And so on. It follows that, for each $n \in \mathbb{N}$,

$$\mu f_1 \ge C\left(P_\alpha f_1 + P_\alpha f_2 + P_\alpha f_3 + \ldots + P_\alpha f_n\right)$$

A simple inductive argument shows that

<4>
$$P_x f_n = \mathbb{P}_x\{\tau_\alpha \ge n\} f_1(\omega_n) = \mathbb{P}_x h_n$$

where

$$h_n(\omega_0, \omega_1, \ldots) = \{n \le \tau_\alpha\} f(\omega_n) = \{\omega_i \notin \alpha \text{ for } 1 \le i < n\} f(\omega_n).$$

Indeed, $P_x f_1 = \mathbb{P}_x f_1(\omega_1) = \mathbb{P}_x h_1$ and

$$\begin{aligned}
P_x f_{n+1} &= P_x^{\omega_1}\{\omega_1 \notin \alpha\} P_{\omega_1} f_n \\
&= \mathbb{P}_x^{\omega_1}\{\omega_1 \notin \alpha\}\mathbb{P}_{\omega_1} h_n \qquad \text{inductive hypothesis} \\
&= \mathbb{P}_x\{\omega_1 \notin \alpha\} h_n(\omega_1, \omega_2, \ldots) \qquad \text{by Markov property} \\
&= \mathbb{P}_x\{\omega_1 \notin \alpha\}\{\omega_i \notin \alpha \text{ for } 2 \le i < n+1\} f(\omega_{n+1}) \\
&= \mathbb{P}_x h_{n+1}
\end{aligned}$$

In particular, $P_\alpha f_i \ge \mathbb{P}_\alpha\{\tau_\alpha \ge i\} f_1(\omega_i)$ for each $i \in \mathbb{N}$, so that

$$\mu f_1 \ge C \sum\nolimits_{1\le i\le n} \mathbb{P}_\alpha\{\tau_a \ge i\} f(\omega_i)$$

Let n tend to ∞ to deduce that $\mu f_1 \ge C\lambda f_1$, as asserted by (ii).

For (iii), first note that equality <3> shows that λ is invariant if $\mathbb{P}_\alpha\{\tau_\alpha = \infty\} = 0$. To establish the reverse implication—that invariance implies recurrence—consider the strictly positive function $f_0(x) = K_x\alpha$, for which <3> and Lemma <1> imply

$$\lambda^x(Pf_0) + \big(P_\alpha f_0\big)\mathbb{P}_\alpha\{\tau_\alpha = \infty\} = \lambda f_0 = \lambda^x K_x\alpha \le \lambda\alpha \le 1.$$

The fact that $\lambda P f_0 = \lambda f_0 \le 1$ and $P_\alpha f_0 > 0$ then implies that $\mathbb{P}\{\tau_\alpha = \infty\} = 0$, that is, α is recurrent, with $1 = \lambda\alpha = \lambda f_0$. If μ is another subinvariant mmeasure with $\mu\alpha = 1$ then

$$1 = \mu\alpha \ge \mu f_0 \ge \lambda f_0 = 1,$$

which forces $\mu f_0 = \lambda f_0$. For each $g \in \mathcal{M}^+(\mathcal{B})$ with $0 \le g \le 1$ we must then have $\mu(f_0 g) = \lambda(f_0 g)$, for otherwise

$$\mu f_0 = \mu f_0 g + \mu f_0(1-g) > \lambda f_0 g + \lambda f_0(1-g) = \lambda f_0.$$

Rescaling then taking monotone limits we then get $\mu(f_0 g) = \lambda(f_0 g)$ for each $g \in \mathcal{M}^+(\mathcal{B})$. Replace g by g/f_0 to conclude that $\mu = \lambda$.

For (iv), we may assume that the finite invariant measure μ has $\mu\alpha = 1$. From $\mu \ge \lambda$ we get $\infty > \lambda\mathcal{X} = \mathbb{P}_\alpha\tau_\alpha =: c^{-1}$ so that α is recurrent and $\lambda = \mu$ by (iii). The probability measure $c\lambda$ is also invariant. $\square$

3. Exercises

Suppose Q is a probability measure concentrated on $\mathbb{N}$. Define a Markov kernel P on $\mathbb{N}$ by

$$P(1, y) = Q\{y\} \qquad \text{for } y \in \mathbb{N}$$

and $P(x, x-1) = 1$ for $x \ge 2$. Define $\sigma_1 = \tau_0 := \inf\{n \in \mathbb{N} : \omega_n = 0\}$ and $\sigma_{i+1} = \inf\{n > \sigma_i : \omega_n = 0\}$.

[1] Show that $\alpha = 0$ in an accessible atom for P.

[2] Suppose Q has finite mean, $\gamma := Q^y y$. Show that the random variables $T_i = \sigma_i - \sigma_{i-1}$ for $i \in \mathbb{N}$ are independent and identically distributed, each with distribution Q. Show also that P has an invariant probability, defined by $\pi\{x\} = Q[x, \infty)/\gamma$.

[3] Suppose $\mathbb{D} \subseteq \mathbb{N}$ is stable under finite sums and $1 = \gcd \mathbb{D}$. Show that $n \in \mathbb{D}$ for all n large enough by following these steps.

(i) First show that there exists an $x \in \mathbb{D}$ such that $x + 1 \in \mathbb{D}$. Start with an arbitrarily chosen pair $x_1 < x_2 = x_1 + r$ from $\mathbb{D}$. If every x in $\mathbb{D}$ is divisible by r then $r = 1$, and we are done. Otherwise there exists some $x \in \mathbb{D}$ such that $x = \alpha r + \gamma = \alpha x_2 - \alpha x_1 + \gamma$ with $\alpha \in \mathbb{N}$ and $0 < \gamma < r$, that is, $\alpha x_2 + \gamma = x + \alpha x_1 \in \mathbb{D}$. If $\gamma > 1$ repeat the argument with x_1 replaced αx_2 and x_2 replaced by $x + \alpha x_1$. And so on.

(ii) If $x, x+1 \in \mathbb{D}$ then $x(x-j) + (x+1)j + kx \in \mathbb{D}$ for $0 \le j \le x$ and $k \in \mathbb{N}_0$. Deduce that $\{x^2 + m : m \in \mathbb{N}_0\} \subseteq \mathbb{D}$.

4. Notes

Theorem <2> is based on Meyn & Tweedie (1993, Theorem 10.2.1). I do not know the history of the result.

References

Meyn, S. P. & Tweedie, R. L. (1993), *Markov Chains and Stochastic Stability*, Springer-Verlag.