1. Notation

Start with a Markov kernel $P(x,\cdot)$ defined for sigma-field $\mathcal{B}$ on a state space $\mathcal{X}$. For $n \in \mathbb{N}$ define the ***n-step transition probabilities*** $P^{(n)}(x,\cdot)$ by $P^{(1)}(x,\cdot) = P(x,\cdot)$ and

$$P^{(n)}(x,\cdot) = P_x^y P^{(n-1)}(y,\cdot)$$

That is, $P^{(n)}(x,f) = \mathbb{P}_x f(X_n)$.

Define a new Markov kernel by

$$K(x,\cdot) = \sum\nolimits_{n\in\mathbb{N}} 2^{-n} P^{(n)}(x,f)$$

so that $K(x,f) = \sum_{n\in\mathbb{N}} 2^{-n}\mathbb{P}_x f(X_n)$. Note that $\mathbb{P}_x\{\tau_A < \infty\} > 0$ if and only if $K(x,A) > 0$.

A set $\alpha \in \mathcal{B}$ is said to be an ***atom*** for the Markov kernel if $P(x,\cdot) = P(x',\cdot)$ for all $x, x' \in \alpha$. Write $\mathbb{P}_\alpha$ for the common distribution $\mathbb{P}_x$ for all $x \in \alpha$. The atom is said to be ***accessible*** if $x \curvearrowright \alpha$ for all $x \in \mathcal{X}$, that is, if $\mathbb{P}_x\{\tau_\alpha < \infty\} > 0$ for all $x \in \mathcal{X}$.

2. Subinvariant measures

A measure λ on $\mathcal{B}$ is said to be ***subinvariant*** for the Markov kernel P if $\lambda^x P_x \le \lambda$, that is, $\lambda^x P_x^y f(y) \le \lambda f$ for all $f \in \mathcal{M}^+(\mathcal{B})$.

If λ is subinvariant, argue inductively that $\lambda^x P^{(n)}(x,f) \le \lambda f$ for each $f \in \mathcal{M}^+(\mathcal{B})$. Then take a weighted sum to deduce that $\lambda^x K(x,f) \le \lambda f$. If α is an accessible atom we have $K(x,\alpha) > 0$ for every $x \in \mathcal{X}$. If $\lambda\alpha < \infty$, it then follows that λ is sigma-fiinite.

<1> **Theorem.** *Suppose α is an accessible atom for the Markov kernel. Define a measure λ on $\mathcal{B}$ by*

$$\lambda f = \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha f(\omega_n)\{n \le \tau_\alpha\} \qquad \text{for each } f \in \mathcal{M}^+(\mathcal{B}).$$

(i) λ is a subinvariant, sigma-finite measure

(ii) λ is invariant if and only if α is recurrent, that is, $\mathbb{P}_\alpha\{\tau_\alpha < \infty\} = 1$.

(iii) $\lambda\mathcal{X} = \mathbb{P}_\alpha \tau_a$

(iv) if ν is a subinvariant measure then $\nu \ge (\nu\alpha)\lambda$.

(v) if there exists a finite subinvariant measure ν with $\nu\alpha > 0$ then $c^{-1} := \mathbb{P}_\alpha\tau_\alpha$ is finite then $\lambda\alpha = 1$, the atom α is recurrent, and $c\lambda$ is an invariant probability measure.

Proof. Note that $Pf(\omega_n) = \mathbb{P}_{\omega_n} f(\omega_{n+1})$. Thus

$$\begin{aligned}
\lambda(Pf) &= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n \le \tau_\alpha\}\mathbb{P}_{\omega_n} f(\omega_{n+1}) \\
&= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n \le \tau_\alpha\} f(\omega_{n+1}) \qquad \text{by Markov property} \\
&= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n = \tau_\alpha\} f(\omega_{\tau+1}) + \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n+1 \le \tau_\alpha\} f(\omega_{n+1}) \\
&= \mathbb{P}_\alpha\{\tau_\alpha < \infty\} f(\omega_{\tau+1}) + \sum\nolimits_{n\ge 2} \mathbb{P}_\alpha\{n \le \tau_\alpha\} f(\omega_n).
\end{aligned}$$

By the strong Markov property, the first term equals $\mathbb{P}_\alpha\{\tau < \infty\} f(\omega_1)$. The last sum is the same as the sum for λf except that the first term, $\mathbb{P}_\alpha f(\omega_1)$, is missing. Add $\mathbb{P}_\alpha\{\tau = \infty\} f(\omega_1)$ to both sides to get

<2>
$$\lambda(Pf) + \mathbb{P}_\alpha\{\tau = \infty\} f(\omega_1) = \lambda f.$$

Clearly λ is subinvariant, and invariant if $\mathbb{P}_\alpha\{\tau_\alpha = \infty\} = 0$.

Sigma-finiteness of λ follows from the fact that

$$\lambda\alpha = \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{\omega_n \in \alpha\}\{n \le \tau_\alpha\} = \mathbb{P}_\alpha\{\tau_\alpha < \infty\} \le 1$$

because $\omega_n \notin \alpha$ for $n < \tau_\alpha$.

To show that invariance implies recurrence, consider the strictly positive function $f(x) = K_x\alpha$, for which <2> implies

$$\lambda^x K_x\alpha + \mathbb{P}_\alpha\{\tau_\alpha = \infty\}K_{\omega_1}\alpha = \lambda^x K_x\alpha \le 1.$$

Deduce that $\mathbb{P}\{\tau_\alpha = \infty\} = 0$, that is, α is recurrent.

The measure λ is finite when

$$\infty > \lambda\mathcal{X} = \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_\alpha\{n \le \tau_\alpha\} = \mathbb{P}_\alpha\tau_\alpha,$$

in which case $\mathbb{P}\{\tau_\alpha = \infty\} = 0$ and λ is invariant.

Next, suppose ν is a subinvariant measure. For a fixed f in $\mathcal{M}^+(\mathcal{B})$ define

$$h_n(\omega_0, \omega_1, \ldots) = \{n \le \tau_\alpha\}f(\omega_n) = \{\omega_i \notin \alpha \text{ for } 1 \le i < n\}f(\omega_n).$$

Define $G_n(x) = \mathbb{P}_x h_n(\omega)$. Note that $G_1(x) = P_x f$ and

$$\begin{aligned} P_x^y\{y \notin \alpha\}G_n(y) &= \mathbb{P}_x\{\omega_1 \notin \alpha\}\mathbb{P}_{\omega_1} h_n \\ &= \mathbb{P}_x\{\omega_1 \notin \alpha\}h_n(\omega_1, \omega_2, \ldots) \qquad \text{by Markov property} \\ &= \mathbb{P}_x\{\omega_1 \notin \alpha\}\{\omega_i \notin \alpha \text{ for } 2 \le i < n+1\}f(\omega_{n+1}) \\ &= G_{n+1}(x). \end{aligned} \tag{<3>}$$

Each G_n takes a constant value, $G_n(\alpha) = \mathbb{P}_\alpha h_n$, on α. Now invoke subinvariance of ν repeatedly:

$$\begin{aligned} \nu f \ge \nu^x P_x f &= \nu^x G_1(x) \\ &= \nu^x\{x \in \alpha\}G_1(x) + \nu^x\{x \notin \alpha\}G_1(x) \\ &\ge CG_1(\alpha) + \nu^x P_x^y\left(\{y \notin \alpha\}G_1(y)\right) \qquad \text{where } C := \nu\alpha \\ &= CG_1(\alpha) + \nu^x\{x \in \alpha\}G_2(x) + \nu^x\{x \notin \alpha\}G_2(x) \qquad \text{by <3>} \\ &\ge C\left(G_1(\alpha) + G_2(\alpha)\right) + \nu^x P_x^y\left(\{y \notin \alpha\}G_1(y)\right). \end{aligned}$$

And so on. For each $n \in \mathbb{N}$,

$$\nu f \ge C\left(G_1(\alpha) + G_2(\alpha) + \ldots + G_n(\alpha)\right)$$

Let n tend to infinity, noting that $\sum_{n\in\mathbb{N}} G_n(\alpha) = \lambda f$ to complete the proof of (iv). □

3. Exercises

Suppose Q is a probability measure concentrated on $\mathbb{N}$. Define a Markov kernel P on $\mathbb{N}$ by

$$P(1, y) = Q\{y\} \qquad \text{for } y \in \mathbb{N}$$

and $P(x, x-1) = 1$ for $x \ge 2$. Define $\sigma_1 = \tau_0 := \inf\{n \in \mathbb{N} : \omega_n = 0\}$ and $\sigma_{i+1} = \inf\{n > \sigma_i : \omega_n = 0\}$.

[1] Show that $\alpha = 0$ in an accessible atom for P.

[2] Suppose Q has finite mean, $\gamma := Q^y y$. Show that the random variables $T_i = \sigma_i - \sigma_{i-1}$ for $i \in \mathbb{N}$ are independent and identically distributed, each with distribution Q. Show also that P has an invariant probability, defined by $\pi\{x\} = Q[x, \infty)/\gamma$.

[3] Suppose $\mathbb{D} \subseteq \mathbb{N}$ is stable under finite sums and $1 = \gcd \mathbb{D}$. Show that $n \in \mathbb{D}$ for all n large enough by following these steps.

(i) First show that there exists an $x \in \mathbb{D}$ such that $x+1 \in \mathbb{D}$. Start with an arbitrarily chosen pair $x_1 < x_2 = x_1 + r$ from $\mathbb{D}$. If every x in $\mathbb{D}$ is divisible by r then $r = 1$, and we are done. Otherwise there exists some $x \in \mathbb{D}$ such that $x = \alpha r + \gamma = \alpha x_2 - \alpha x_1 + \gamma$ with $\alpha \in \mathbb{N}$ and $0 < \gamma < r$, that is, $\alpha x_2 + \gamma = x + \alpha x_1 \in \mathbb{D}$. If $\gamma > 1$ repeat the argument with x_1 replaced αx_2 and x_2 replaced by $x + \alpha x_1$. And so on.

(ii) If $x, x+1 \in \mathbb{D}$ then $x(x-j) + (x+1)j + kx \in \mathbb{D}$ for $0 \le j \le x$ and $k \in \mathbb{N}_0$. Deduce that $\{x^2 + m : m \in \mathbb{N}_0\} \subseteq \mathbb{D}$.

4. Notes

Theorem <1> is based on Meyn & Tweedie (1993, Theorem 10.2.1). I do not know the history of the result.

References

Meyn, S. P. & Tweedie, R. L. (1993), *Markov Chains and Stochastic Stability*, Springer-Verlag.