The multivariate normal distribution

0.1 Basic facts

Let $Z_1, Z_2, \ldots, Z_n$ be independent $N(0,1)$ random variables. When treated as the coordinates of a point in $\mathbb{R}^n$ they define a random vector $\mathbf{Z}$, whose (joint) density function is

$$f(\mathbf{z}) = (2\pi)^{-n/2} \exp\left(-\tfrac{1}{2}\sum\nolimits_i z_i^2\right) = (2\pi)^{-n/2} \exp\left(-\tfrac{1}{2}\|\mathbf{z}\|^2\right).$$

Such a random vector is said to have a ***spherical normal distribution***.

The ***chi-square***, χ_n^2, is defined as the distribution of the sum of squares $R^2 := Z_1^2 + \cdots + Z_n^2$ of independent $N(0,1)$ random variables. The ***noncentral chi-square***, $\chi_n^2(\gamma)$, with noncentrality parameter $\gamma \geq 0$ is defined as the distribution of the sum of squares $(Z_1+\gamma)^2 + Z_2 \cdots + Z_n^2$.

The random vector $\mathbf{Z}/R$ has length 1; it takes values on the unit sphere $S := \{\mathbf{z} \in \mathbb{R}^n : \sum_{i\leq n} z_i^2 = 1\}$. By symmetry of the joint density $f(\mathbf{z})$, the random vector is uniformly distributed on S, no matter what value R takes. In other words $\mathbf{Z}/R$ is independent of R. This fact suggests a way to construct a random vector with the same distribution as $\mathbf{Z}$: Start with a random variable T^2 that has a χ_n^2 distribution independent of a random vector $\mathbf{U}$ that is uniformly distributed on the unit sphere S. Then the components of the random vector $T\mathbf{U}$ are independent $N(0,1)$'s. In two dimensions, the random vector $\mathbf{U}$ can be defined by

$$\mathbf{U} = (\cos V, \sin V) \qquad \text{where } V \sim \text{Unif}(0, 2\pi].$$

0.2 New coordinate system

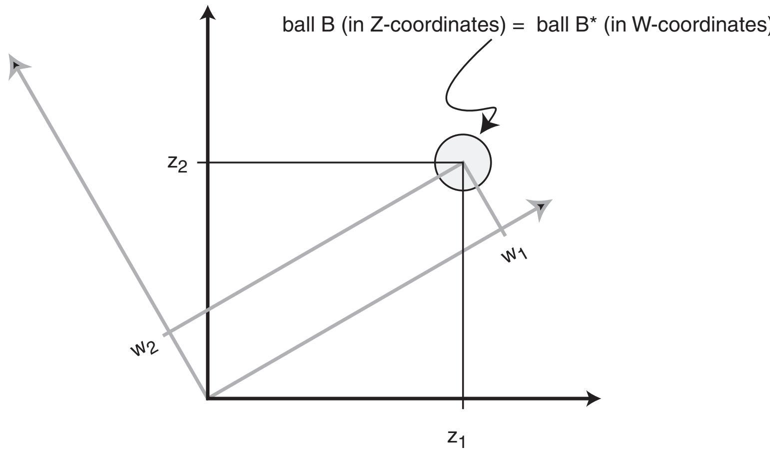
The spherical symmetry of the density $f(\cdot)$ is responsible for an important property of multivariate normals. Let $\mathbf{q}_1, \ldots, \mathbf{q}_n$ be a new orthonormal basis for $\mathbb{R}^n$, and let

$$\mathbf{Z} = W_1\mathbf{q}_1 + \cdots + W_n\mathbf{q}_n$$

be the representation for $\mathbf{Z}$ in the new basis.

<1> **Theorem.** *The $W_1, \ldots, W_n$ are also independent $N(0,1)$ distributed random variables.*

If you know about multivariate characteristic functions this is easy to establish using the matrix representation $\mathbf{Z} = Q\mathbf{W}$, where Q is the orthogonal matrix with columns $\mathbf{q}_1, \dots, \mathbf{q}_n$.



A more intuitive explanation is based on the approximation

$$\mathbb{P}\{\mathbf{Z} \in B\} \approx f(\mathbf{z})(\text{volume of } B)$$

for a small ball B centered at $\mathbf{z}$. The transformation from $\mathbf{Z}$ to $\mathbf{W}$ corresponds to a rotation, so

$$\mathbb{P}\{\mathbf{Z} \in B\} = \mathbb{P}\{\mathbf{W} \in B^*\},$$

where B^* is a ball of the same radius, but centered at the point $\mathbf{w} = (w_1, \dots, w_n)$ for which $w_1\mathbf{q}_1 + \cdots + w_n\mathbf{q}_n = \mathbf{z}$. The last equality implies $\|\mathbf{w}\| = \|\mathbf{z}\|$, from which we get

$$\mathbb{P}\{\mathbf{W} \in B^*\} \approx (2\pi)^{-n/2} \exp(-\tfrac{1}{2}\|\mathbf{w}\|^2)(\text{volume of } B^*).$$

That is, $\mathbf{W}$ has the asserted spherical normal density.

To prove results about the spherical normal it is often merely a matter of transforming to an appropriate orthonormal basis.

<2> **Theorem.** *Let $\mathcal{X}$ be an m-dimensional subspace of $\mathbb{R}^n$. Let $\mathbf{Z}$ be a vector of independent $N(0,1)$ random variables, and $\boldsymbol{\mu}$ be a vector of constants. Then*

(i) the projection $\widehat{\mathbf{Z}}$ of $\mathbf{Z}$ onto $\mathcal{X}$ is independent of the projection $\mathbf{Z} - \widehat{\mathbf{Z}}$ of $\mathbf{Z}$ onto $\mathcal{X}^\perp$, the orthogonal complement of $\mathcal{X}$.

(ii) $\left\|\widehat{\mathbf{Z}}\right\|^2$ has a χ^2_m distribution.

(iii) $\|\mathbf{Z}+\boldsymbol{\mu}\|^2$ *has a noncentral* $\chi^2_n(\gamma)$ *distribution, with* $\gamma = \|\boldsymbol{\mu}\|$.

(iv) $\left\|\widehat{\mathbf{Z}}+\boldsymbol{\mu}\right\|^2$ *has a noncentral* $\chi^2_m(\gamma)$ *distribution, with* $\gamma = \|\boldsymbol{\mu}\|$.

PROOF Let $\mathbf{q}_1, \dots \mathbf{q}_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $\mathbf{q}_1, \dots, \mathbf{q}_m$ span the space $\mathcal{X}$ and $\mathbf{q}_{m+1}, \dots, \mathbf{q}_n$ span $\mathcal{X}^\perp$. If $\mathbf{Z} = W_1\mathbf{q}_1 + \dots + W_n\mathbf{q}_n$ then

$$\begin{aligned}\widehat{\mathbf{Z}} &= W_1\mathbf{q}_1 + \dots + W_m\mathbf{q}_m,\\ \mathbf{Z}-\widehat{\mathbf{Z}} &= W_{m+1}\mathbf{q}_{m+1} + \dots + W_n\mathbf{q}_n,\\ \|\mathbf{Z}\|^2 &= W_1^2 + \dots + W_m^2,\end{aligned}$$

from which the first two asserted properties follow.

For the third and fourth assertions, choose the basis so that $\boldsymbol{\mu} = \gamma\mathbf{q}_1$. Then

$$\begin{aligned}\mathbf{Z}+\boldsymbol{\mu} &= (W_1+\gamma)\mathbf{q}_1 + W_2\mathbf{q}_2 + \dots \qquad\qquad + W_n\mathbf{q}_n\\ \widehat{\mathbf{Z}}+\boldsymbol{\mu} &= (W_1+\gamma)\mathbf{q}_1 + W_2\mathbf{q}_2 + \dots + W_m\mathbf{q}_m\end{aligned}$$

from which we get the noncentral chi-squares.

□

0.3 Fact about the general multivariate normal

If Z is an $n \times 1$ vector of independent $N(0,1)$ random variables, if μ is an $m \times 1$ vector of constants, and if A is an $m \times n$ matrix of constants, then the random vector $X = \mu + AZ$ has expected value μ and variance matrix

$$V = \mathbb{E}(X-\mu)(X-\mu)' = \mathbb{E}(AZZ'A') = A\mathbb{E}(ZZ')A' = AA'.$$

The moment generating function of Z is defined as

$$\begin{aligned}M_Z(s) &= \mathbb{E}\exp(s'Z) = \mathbb{E}\left(e^{s_1Z_1}\dots e^{s_nZ_n}\right)\\ &= \prod\nolimits_{i\le n}\mathbb{E}e^{s_jZ_j} \qquad \text{by independence}\\ &= \exp(\sum\nolimits_j s_j^2) = \exp(|s|^2).\end{aligned}$$

The moment generating function for the random vector X is defined as

$$\begin{aligned}M_X(t) &= \mathbb{E}\exp(t'X)\\ &= \mathbb{E}\exp(t'\mu + t'AZ) = e^{t'\mu}M_Z(A't)\\ &= \exp(t'\mu + t'AA't/2) = \exp(t'\mu + t'Vt/2) \qquad \text{for } t \in \mathbb{R}^m.\end{aligned}$$

In particular, the distribution of X depends only on μ and V. The random vector X has a $N(\mu, V)$ distribution. If V is nonsingular, the distribution has a multivariate density

$$f(x) = (2\pi \det(V))^{-m/2} \exp\left(\tfrac{1}{2}(x-\mu)' V^{-1}(x-\mu)\right) \qquad \text{for } x \in \mathbb{R}^m.$$

If V is singular, the distribution of $X - \mu$ concentrates in some lower-dimensional subspace of $\mathbb{R}^m$; it is no longer determined by the density.

If γ is a $k \times 1$ vector of constants and B is a $k \times m$ matrix of constants then

$$\gamma + BX = (\gamma + B\mu) + BAZ$$

has a $N(\gamma + B\mu, BVB')$ distribution.

0.4 Independence

Suppose the $N(\mu, V)$ distributed random vector X is thought of as the concatenation of two subvectors: an $m_1 \times 1$ vector X_1 and an $m_2 \times 1$ vector X_2, with corresponding decompositions

$$\begin{aligned} \mu &= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\ V &= \begin{pmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{pmatrix} = \mathbb{E}\begin{pmatrix} (X_1-\mu_1)(X_1-\mu_1)' & (X_1-\mu_1)(X_2-\mu_2)' \\ (X_2-\mu_2)(X_1-\mu_1)' & (X_2-\mu_2)(X_2-\mu_2)' \end{pmatrix}. \end{aligned}$$

Then $V_{j,j} = \text{var}(X_j)$ and $V_{1,2} = V_{2,1}'$ is the $m_1 \times m_2$ matrix $\text{cov}(X_1, X_2)$.

The joint moment generating function becomes

$$\begin{aligned} M_{x_1,X_2}(t_1, t_2) &= M_X(t) \qquad \text{where } t' = (t_1', t_2') \\ &= \exp\left(t_1'\mu_1 + t_2'\mu_2 + \tfrac{1}{2}t_1'V_{1,1}t_1 + t_1'V_{1,2}t_2 + \tfrac{1}{2}t_2'V_{2,2}t_2\right), \end{aligned}$$

which factorizes for all t if and only if $V_{1,2} = 0$. Factorization of the joint moment generating function for all t is the necessary and sufficient condition for X_1 and X_2 to be independent.

Joint normality is one of the rare situations where independence is equivalent to zero covariance.

0.5 Conditional distributions

Suppose X_1 and X_2 have a joint normal distribution, that is, the distribution of the column vector $[X_1, X_2]$ is multivariate normal. For each $m_1 \times m_2$ matrix B,

$$\begin{aligned}
&\mathbb{E}\exp(s_1'(X_1 - BX_2) + s_2'X_2) \\
&\qquad = \exp\left(s_1'X_1 + (s_2 - Bs_1)'X_2\right) \\
&\qquad = M_{X_1,X_2}(s_1, s_2 - Bs_1) \\
&\qquad = \exp\left(s_1'\mu_1 + (s_2 - B's_1)'\mu_2 + \tfrac{1}{2}s_1'V_{1,1}s_1 + \dots\right).
\end{aligned}$$

If you write out all the terms in the last line you will get a quadratic in s_1 and s_2. The joint distribution of $X_1 - BX_2$ and X_2 is bivariate normal with covariance matrix

$$\operatorname{cov}(X_1 - BX_2, X_2) = V_{1,2} - BV_{2,2}.$$

If $V_{2,2}$ is nonsingular and we choose $B = V_{1,2}V_{2,2}^{-1}$ then the covariance matrix becomes zero. In that case $X_1 - BX_2$ and X_2 are independent. In particular, the conditional distribution of $X_1 - BX_2$ given X_2 is the same as the unconditional distribution of $X_1 - BX_2$, which is multivariate normal with expected value $\mu_1 - B\mu_2$ and variance matrix for which

$$\operatorname{var}(X_1) = \operatorname{var}(X_1 - BX_2) + \operatorname{var}(BX_2).$$

That is,

$$X_1 - BX_2 \mid X_2 = x_2 \sim N(\mu_1 - B\mu_2, V_{1,1} - BV_{2,2}B').$$

Put another way,

$$X_1 - \mu_1 \mid X_2 = x_2 \sim N(V_{1,2}V_{2,2}^{-1}(x_2 - \mu_2), V_{1,1} - V_{1,2}V_{2,2}^{-1}V_{2,1}).$$

<3> **Example.** Specialize the last result to the bivariate normal case, where $m_1 = m_2 = 1$ and

$$V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

The conditonal distribution of X_1 given $X_2 = x_2$ is

$$N\left(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2 - \mu_2), \sigma_1^2(1 - \rho^2)\right).$$

□

0.6 Standard distributions

Suppose

Z has a $N(0,1)$ distribution
S_k^2 has a χ_k^2 distribution
S_ℓ^2 has a χ_ℓ^2 distribution

with all random variables independent of each other. Then, by definition,

$$\frac{Z}{\sqrt{S_k^2/k}} \text{ has a } t\text{-distribution on } k \text{ degrees of freedom } (t_k)$$

and

$$\frac{S_\ell^2/\ell}{S_k^2/k} \text{ has an } F\text{-distribution on } \ell \text{ and } k \text{ degrees of freedom } (F_{\ell,k})$$

The picture shows the t_k-densities for $k = 1, \dots, 6$ with the $N(0,1)$ density superimposed (dotted line). The height of the density at 0 increases with k.