Statistics 610 fall 2014
Homework # 1
Due: Thursday 11 September

[1.1] Suppose $T : \mathbb{R} \to \mathbb{R}^k$ with $T(z) = [T_1(z), \dots, T_k(z)]$. Let f_0 be a nonnegative function on $\mathbb{R}$.

(i) Define Θ as the set of $\theta = [\theta_1, \dots, \theta_k] \in \mathbb{R}^k$ for which

$$\int \exp(\theta \cdot T(z)) f_0(z)\, dz < \infty.$$

Use the Hölder inequality (see below) to show that Θ is convex.

(ii) Show that the function Ψ defined on Θ by

$$e^{\Psi(\theta)} = \int \exp(\theta \cdot T(z)) f_0(z)\, dz$$

is convex. Hint: Consider the line $\theta_r = (1-r)\theta_0 + r\theta_1$ for $0 \le r \le 1$, for given θ_0, θ_1 in Θ. Calculate derivatives (with respect to r) of $\exp(\Psi(\theta_r))$.

[1.2] Consider the maximum likelihood estimator based on the model $\mathbb{P}_\theta$: the data $x_1, \dots, x_n$ are independent observations from the discrete distribution with $f_\theta(z) = \theta^z(1-\theta)^{1-z}$ for $z \in \{0, 1\}$, and $\theta \in [0, 1]$. (Independent coin tosses with probability θ of heads.) In class I defined the functions $G_n(t)$ and $G_\theta(t)$, and gave a quadratic approximation to G_n under $\mathbb{P}_\theta$. Draw pictures (computer generated) that illustrate the fact that $G_n \approx G_\theta$ for n large (how large?); and near θ, the function G_n is well approximated by a particular quadratic when n is large enough. Give details (code) for how you produce the pictures and calculate the approximations.

[1.3] Suppose f and g are probability density functions on the real line. Show that

$$\int_{-\infty}^{+\infty} f(z) \log\left(f(z)/g(z)\right)\, dz \ge \int_{-\infty}^{+\infty} \left(\sqrt{f(z)} - \sqrt{g(z)}\right)^2 dz$$

by arguing as follows. Define $\Delta(z)$ by $\sqrt{f(z)} = \sqrt{g(z)}(1 + \Delta(z))$. Show that

$$\int f \log(f/g) \ge 2 \int g(1+\Delta)\Delta = 2 \int \sqrt{f}\left(\sqrt{f} - \sqrt{g}\right).$$

Then use the fact that

$$\int \left(\sqrt{f} - \sqrt{g}\right)^2 = 2 - 2 \int \sqrt{fg}.$$

Hölder's inequality
Suppose F and G are nonnegative functions for which $\int F f_0 < \infty$ and $\int G f_0 < \infty$. For each r in $(0,1)$,

$$\int F^r G^{1-r} f_0 \leq \left(\int F f_0\right)^r \left(\int G f_0\right)^{1-r}$$

Proof Check that both sides of the equality are divided by C^r if F is replaced by F/C. Choose $C = \int F f_0$ to see that, without loss of generality, $\int F f_0 = 1$. Similarly, without loss of generality, suppose $\int G f_0 = 1$. For the integrand on the left-hand side note that

$$\begin{aligned} F^r G^{1-r} &= \exp\left(r \log F + (1-r)\log G\right) \\ &\leq \exp\left(\log(rF + (1-r)G)\right) \qquad \text{by concavity of the log function} \\ &= rF + (1-r)G. \end{aligned}$$

Multiply both sides by f_0 then integrate to deduce that

$$\int F^r G^{1-r} f_0 \leq \int \left(rF + (1-r)G\right) f_0 = 1.$$

□