Statistics 610 fall 2014
Homework # 2
Due: Thursday 18 September

[2.1] In my R script mleCauchy.R (in the Handouts directory) I tried to calculate the quadratic approximation, $tZ_n/\sqrt{n} - \frac{1}{2}t^2\mathbb{I}(0)$, to the log-likelihood for the Cauchy location model, with densities

$$f_t(z) = f_0(z-t) \qquad \text{where } f_0(z) = \frac{1}{\pi(1+z^2)}$$

and $t \in \mathbb{R}$. I took observations $x_1, \dots, x_n$ from f_0.

(i) Find Z_n and calculate the Fisher information function $\mathbb{I}(t)$ for the model. (If you doing everything manually, you'll probably end up making substitutions $z = \tan(y)$ and then have to calculate integrals of polynomials in $\sin(y)$ and $\cos(y)$. I found it easiest to reexpress everything as a polynomial in e^{iy}.)

(ii) What is the approximate distribution of the MLE for the Cauchy location model?

[2.2] Suppose M_n is the median of a sample $x_1, \dots, x_n$ from a P with (continuous) distribution function F and density f. (You may assume that n is odd, $n = 2k+1$, so that there is no messing around with the definition of sample median.) For a fixed y, write N_y for $\sum_{i\le n} \mathbf{1}\{x_i \le y\}$, the number of observations in the interval $(-\infty, y]$.

(i) Explain why $\{M_n \le y\} = \{N_y \ge k+1\}$.

(ii) Explain why $(N_y - nF(y))/\sqrt{nF(y)(1-F(y))}$ has approximately a standard normal distribution.

(iii) Let m be the population median, that is, $F(m) = 1/2$. Taylor gives

$$F(m + w/\sqrt{n}) \approx 1/2 + wf(m)/\sqrt{n}.$$

Use this expansion, together with the ideas from parts (i) and (ii), to show that $\sqrt{n}(M_n - m)$ is approximately normally distributed. Hint: Start with $\mathbb{P}\{\sqrt{n}(M_n - m) \le r\}$.

(iv) For the Cauchy location problem, find the approximate distribution of the sample median. According to Fisher, the variance in the approximating normal should be greater than the corresponding variance for the MLE.

[2.3] (Hard; extremely optional) In class on September 11 you saw lots of technical details that are usually hidden in a course at this level. In particular, you saw the role played by the process $Z_n(t) = n^{-1/2}\sum_{i\le n} \dot{g}(x_i, t)$ for t near θ. Define $H_\theta(t) = \mathbb{E}_\theta \dot{g}(z, t)$ and

$$W_n(t) = Z_n(t) - \mathbb{E}_\theta Z_n(t) = n^{-1/2}\sum_{i\le n}\left(\dot{g}(x_i, t) - H_\theta(t)\right).$$

For each constant C show (under the assumptions described in class) that

$$\mathbb{E}_\theta \sup_{|h|\le C} |W_n(\theta + h/\sqrt{n}) - W_n(\theta)| \to 0 \qquad \text{as } n \to \infty.$$

What does this tell you about the behavior of $Z_n(\theta_n^*)$ (under $\mathbb{P}_\theta$) if θ_n^* is a root-n consistent estimator for θ? Hint: Use more Taylor, starting from

$$\sqrt{n}\left(Z_n(\theta+\delta) - Z_n(\theta)\right) = \sum_{i\le n} \delta \int_0^1 \ddot{g}(x_i, \theta + s\delta)\, ds.$$