**Statistics 241/541 fall 2014**
**2011: Solutions to sheet 1**

*[1]* *(5 points) Reconsider the problem where: there is a prize in one of boxes $1, 2, 3$; contestant Sam chooses a box; then the host reveals that there is no prize in one of the other boxes. Suppose the host has a preference for smaller numbers: when faced with a choice between boxes $i$ and $j$ (neither containing the money) he chooses the smaller number with probability $2/3$. Explain what Sam's strategy should be if he chooses box 1 then the host reveals that box 2 is empty.*

For $i = 1, 2, 3$ define

$$M_i = \{\text{box } i \text{ contains money}\}$$
$$S_i = \{\text{Sam chooses box } i\}$$
$$H_i = \{\text{Host reveals box } i \text{ is empty}\}$$

I interpret the problem to mean $\mathbb{P}S_i = \mathbb{P}(S_i \mid M_j) = p_i$ for all $i$ and $j$ and

$$\mathbb{P}(H_2 \mid S_1 M_1) = 2/3$$
$$\mathbb{P}(H_2 \mid S_1 M_2) = 0$$
$$\mathbb{P}(H_2 \mid S_1 M_3) = 1$$

Then

$$\begin{aligned}\mathbb{P}(M_1 \mid S_1 H_2) &= \frac{\mathbb{P}(M_1 S_1 H_2)}{\mathbb{P}(S_1 H_2)} \\ &= \frac{\mathbb{P}(S_1)\mathbb{P}(M_1 \mid S_1)\mathbb{P}(H_2 \mid S_1 M_1)}{\sum_i \mathbb{P}(S_1)\mathbb{P}(M_i \mid S_1)\mathbb{P}(H_2 \mid S_1 M_i)} \\ &= \frac{p_1 \times {}^1\!/_3 \times {}^2\!/_3}{(p_1 \times {}^1\!/_3 \times {}^2\!/_3) + 0 + (p_1 \times {}^1\!/_3 \times 1)} = \frac{2}{5}\end{aligned}$$

Sam should switch to box 3. Notice that the $p_i$'s have no effect on the answer (provided $p_1 > 0$).

*[2]* *(5 points) Roll two unfair dice: one has the 5 replaced by a second 3, the other has the numbers $0, 2, 3, 3, 4, 5$. (Apart from the strange numbers, each of the six sizes is equally likely to end face up.) Let $Y$ denote the number showing*

*on the first die, and $Z$ denote the number showing on the second die. Under the assumption that each event determined only by the first die is independent of each event determined only by the second die, the events $A = \{Y \text{ is odd}\}$ and $B = \{Z \text{ is odd}\}$ are (by assumption) independent. Show $A$ and $B$ are not conditionally independent given the event $E = \{Y + Z \text{ is even}\}$.*

Notice that $E = AB \cup A^cB^c$ and $\mathbb{P}A = 1/2$ and $\mathbb{P}B = 1/2$, so that

$$\mathbb{P}E = \mathbb{P}(AB) + \mathbb{P}(A^cB^c) = (1/2 \times 1/2) + (1/2 \times 1/2) = 1/2.$$

The probabilities of $AB$ and of $A^cB^c$ factorize because $A$ and $B$ are independent. As $ABE = AB$, we then have

$$\mathbb{P}(AB \mid E) = \frac{\mathbb{P}(AB)}{\mathbb{P}E} = \frac{1/2 \times 1/2}{1/2} = \frac{1}{2}$$

whereas

$$\mathbb{P}(A \mid E) \times \mathbb{P}(B \mid E) = \frac{\mathbb{P}(AE) \times \mathbb{P}(BE)}{(\mathbb{P}E)^2} = \frac{\mathbb{P}(AB) \times \mathbb{P}(BA)}{(\mathbb{P}E)^2} = \frac{1}{4}.$$

Alternatively (which I prefer),

$$\mathbb{P}(A \mid BE) = \mathbb{P}(A \mid AB) = 1 \qquad \text{because } BE = BAB \cup BA^cB^c = AB \subset A$$

whereas

$$\mathbb{P}(A \mid B) = \mathbb{P}A = 1/2 \qquad \text{because } A \text{ and } B \text{ are independent.}$$

In fact, as shown by the calculations given above, the events $A$ and $B$ are independent, the events $A$ and $E$ are independent, and the events $B$ and $E$ are independent. (This situation is sometimes referrred to as pairwise independence.) But $AB$ is not independent of $E$:

$$\mathbb{P}(AB) = 1/4 \neq 1/2 = \mathbb{P}(AB \mid E).$$

Even better, $\mathbb{P}(E \mid AB) = 1 \neq \mathbb{P}E$.

Some of you treated $E$ as the "info". It is better to write $E$ when the info is so specific. You could perhaps take the info as: independence of the dice and the assumption that each face of a die has probability $1/6$ of showing. I omitted all mention of info, but perhaps it would be better to assert that $A$ and $B$ are independent given that info, and so on.