Normal approximation to the Binomial

Talk by David Pollard at the Orsay meeting in honor of

Jean Bretagnolle
Didier Dacunha-Castelle
and
Ildar Ibragimov

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The quantile coupling between the Bin(n, $\frac{1}{2}$) and the $N(n/2, n/4)$ distributions has played a surprisingly important role in probability and statistics. For example, it is the basic ingredient in the KMT coupling of the empirical process with a Brownian Bridge. More recently, it has emerged as a tool for bounding the Le Cam distance between density and white noise models.

The talk will sketch some applications and explain a simple method for deriving bounds for the quantile coupling.

http://www.stat.yale.edu/~pollard/Paris2001

$$\text{Bin}(n, \tfrac{1}{2}) \approx N\left(\frac{n}{2}, \frac{n}{4}\right)$$

De Moivre (1733)

⋮ ⋮

Peizer & Pratt (1968); Molenaar (1970)

⋮

Komlós, Major & Tusnády (1975)

⋮

Tusnády (1977)

⋮

Csörgő & Révész (1981, Section 4.4)

⋮

Bretagnolle & Massart (1989)

⋮

Carter & Pollard (2000)

Tusnády bound

There exist $X \sim \text{Bin}(n, \frac{1}{2})$ and $Y \sim N(n/2, n/4)$ with

$$|X - n/2| \leq |Y - n/2| + 1$$

and

$$|X - Y| \leq 1 + \frac{(Y - n/2)^2}{2n}$$

Note:

$$X - \frac{n}{2} \text{ is of order } \sqrt{n}$$
$$Y - \frac{n}{2} \text{ is of order } \sqrt{n}$$

Tusnády: $X - Y$ is of order $O_p(1)$

Coupling empirical process with Brownian Bridge

$P =$ Lebesgue measure on $[0, 1]$.

Haar basis for $L^2(P)$:

$$\Psi = \{1\} \cup \{\psi_{i,k} : 0 \le i < 2^k,\ k \in \mathbb{N}_0\}$$
$$f - Pf = \sum_{k=0}^{\infty} \sum_i \psi_{i,k} \langle f, \psi_{i,k} \rangle,$$

Brownian Bridge:

$$\nu(f) = \sum_{k=0}^{\infty} \sum_i \eta_{i,k} \langle f, \psi_{i,k} \rangle$$
with $\eta_{i,k} = \nu(\psi_{i,k})$ independent $N(0, 1)$.

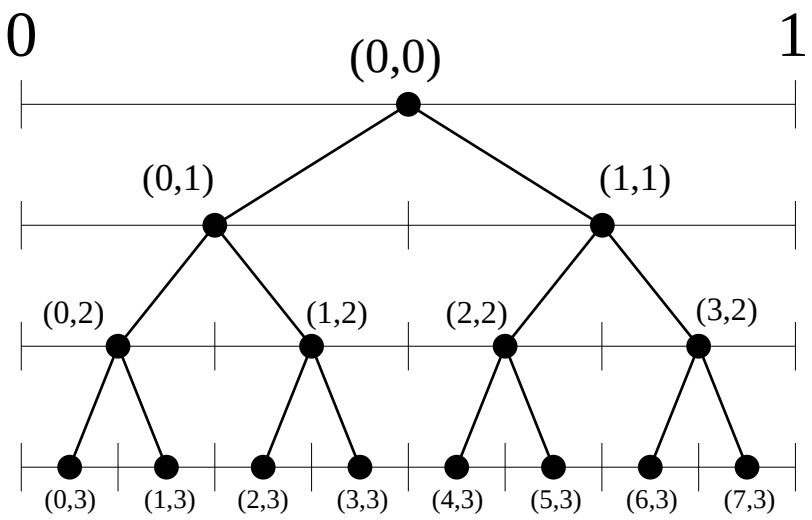
Empirical process $\nu_n := \sqrt{n}(P_n - P)$:

$$\nu_n(f) \stackrel{?}{=} \sum_{k=0}^{\infty} \sum_i \nu_n(\psi_{i,k}) \langle f, \psi_{i,k} \rangle.$$

Use Tusnády to couple $\nu_n(\psi_{i,k})$ with $\eta_{i,k}$.

KMT: $f \in \{[0, t] : 0 \le t \le 1\}$

cf. UGMTP §10.6



Contribution from coupling down a path

$$(i_0, 0),\ (i_1, 1), (i_2, 2) \ldots, (i_m, m) :$$

$$\sum_{k=0}^{m} 2^{-k/2} |\nu_n(\psi_{i_k,k}) - \eta_{i_k,k}| \leq \frac{C}{\sqrt{n}} \sum_{k=0}^{m} \left(1 + \eta_{i_k,k}^2\right)$$

Quantile transformation

Find

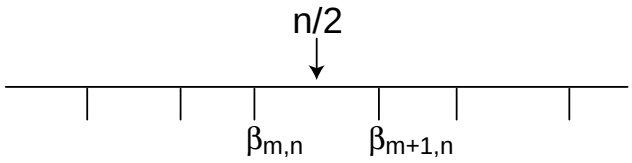
$$-\infty = \beta_{0,n} < \beta_{1,n} < \ldots < \beta_{n,n} < \beta_{n+1,n} = \infty$$

with

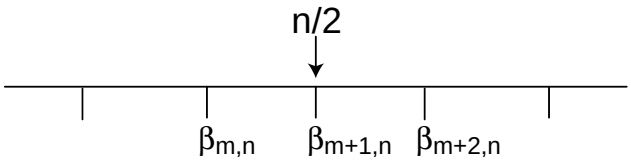
$$\mathbb{P}\left\{\text{Bin}(n, \tfrac{1}{2}) \geq k\right\} = \mathbb{P}\left\{N\left(\frac{n}{2}, \frac{n}{4}\right) > \beta_{k,n}\right\}.$$

Put

$$X = k \qquad \text{when } \beta_{k,n} \leq Y < \beta_{k+1,n}$$

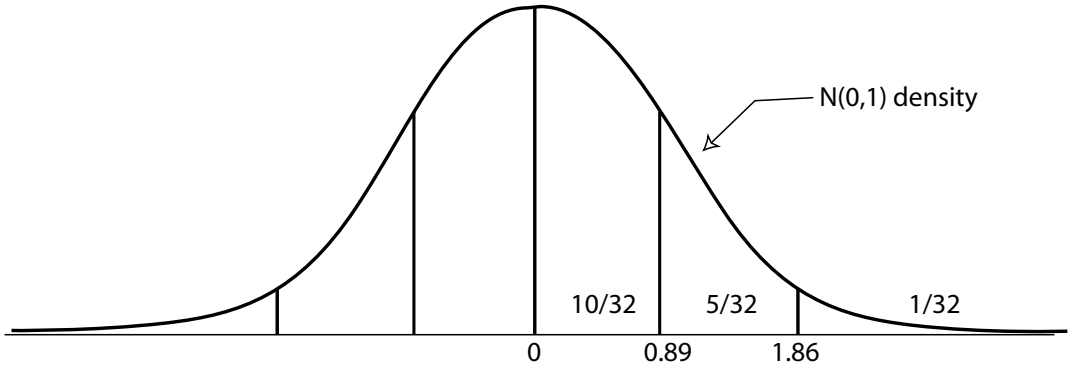


n = 2m, even



n = 2m+1, odd

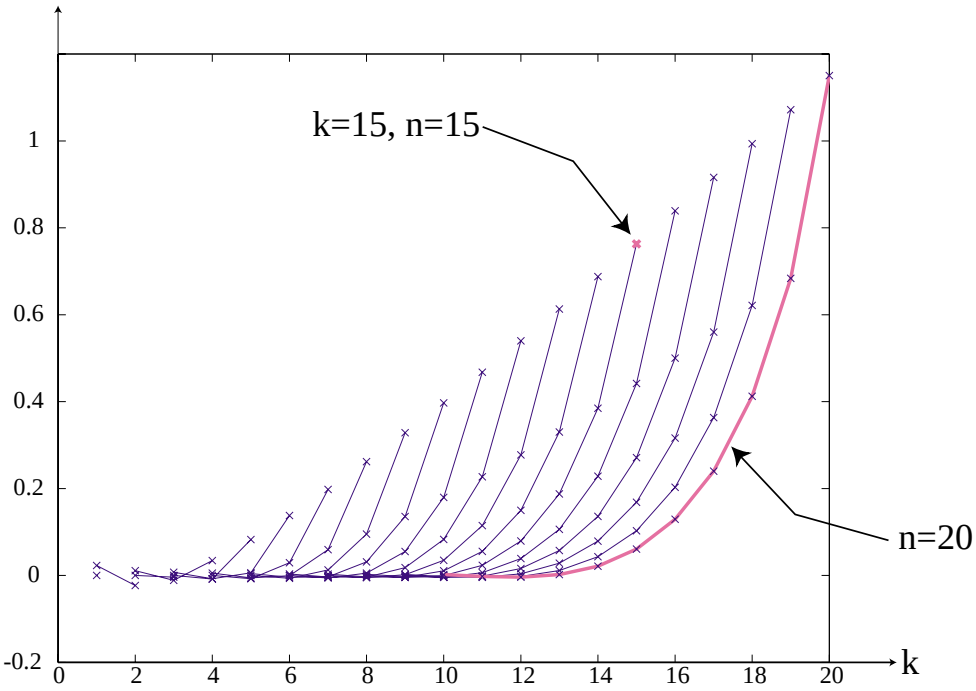
Bin(5, 1/2) EXAMPLE



N(5/2,5/4) cutpoints:

0.42 1.51 2.50 3.49 4.58

$\beta_{k,n} - (k - 1/2)$ versus k
for
$n/2 \leq k \leq n$ and $1 \leq n \leq 20$



$\mathbb{P}\{X \geq k\} \approx \mathbb{P}\{Y \geq k - \frac{1}{2}\}$ and $\beta_{k,n} \approx k - \frac{1}{2}$?

Approximations for $k > n/2$

Tusnády, via Bretagnolle & Massart (1989, Appendix):

$$k - 1 \le \beta_{k,n} \le 3n/2 - n\sqrt{1-\epsilon_k} \qquad \text{with } \epsilon_k := (2k-n)/n$$
$$= k + \frac{(k-n/2)^2}{2n} + \ldots \qquad \text{for } k \ll n$$

Carter & Pollard (2000):

$$k - \tfrac{1}{2} + f_{k,n} - \frac{C}{\sqrt{n}} \le \beta_{k,n} \le k - \tfrac{1}{2} + f_{k,n} + \frac{C\log n}{\sqrt{n}}$$

with

$$f_{k,n} = \frac{(k-n/2)^3}{3n^2} + \ldots \qquad \text{for } k \ll n$$

Compare when $|k - n/2| = o(n^{2/3})$ or $k \approx n$.

Extreme tail, for fixed B

$$\beta_{n-B,n} = \frac{1+c}{2}n - \frac{1+2B}{4c}\log n + O(1)$$

where

$$c = \sqrt{2\log 2} \approx 1.177 \qquad \text{and} \qquad \frac{1+c}{2} \approx 1.088$$

Tusnády:

$$\beta_{n-B,n} \leq \frac{3n}{2} - \sqrt{2nB}$$

THEOREM

Standardized cutpoints:

$$z_{k,n} := 2(\beta_{k,n} - n/2)/\sqrt{n}$$
$$u_{k,n} := 2\left(k - 1/2 - n/2\right)/\sqrt{N} \qquad \text{where } N := n - 1.$$

Uniformly in $(n+1)/2 \le k < n$,

$$\boxed{z_{k,n} = u_{k,n} S\left(u_{k,n}/\sqrt{N}\right) + \frac{\log\left(1 - u_{k,n}^2/N\right)}{2c|u_{k,n}|} + R_{k,n}}$$

with

$$\boxed{-O\left(\frac{|u_{k,n}|+1}{n}\right) \le R_{k,n} \le O\left(\frac{|u_{k,n}|+\log n}{n}\right)}$$

where $S(\epsilon) := \sqrt{1 + 2\epsilon^2\gamma(\epsilon)}$, for the increasing function $\gamma(\cdot)$ defined for $0 < \epsilon < 1$ by

$$\gamma(\epsilon) := \frac{(1+\epsilon)\log(1+\epsilon) + (1-\epsilon)\log(1-\epsilon) - \epsilon^2}{2\epsilon^4}$$
$$= \sum_{r=0}^{\infty} \frac{\epsilon^{2r}}{(2r+3)(2r+4)}.$$

Note: $\gamma(0) = 1/12$ and $1 + 2\gamma(1) = c^2 := 2\log 2$.

Method of proof

$$\mathbb{P}\{X \geq k\} = \frac{n!}{(k-1)!(n-k)!} \int_0^{1/2} t^{k-1}(1-t)^{n-k}\, dt.$$

Put $N := n-1$ and $K := k-1$ and

$$\alpha := \frac{1+\epsilon}{2} := K/N := 1-\overline{\alpha}.$$
$$H(t) := \alpha \log t + \overline{\alpha} \log(1-t)$$

Stirling:

$$\frac{n!}{(k-1)!(n-k)!} \approx \sqrt{\frac{N}{2\pi\alpha\overline{\alpha}}} \exp(-NH(\alpha))$$

$$\begin{aligned}\mathbb{P}\{X \geq k\} &\approx \sqrt{\frac{N}{2\pi\alpha\overline{\alpha}}} \int_0^{1/2} \exp\left(NH(t) - NH(\alpha)\right)\, dt \\ &\approx \sqrt{\frac{N}{2\pi\alpha\overline{\alpha}}} \int_0^{1/2} \exp\left(-N\epsilon^4\gamma(\epsilon) - 2N\left(s+\frac{\epsilon}{2}\right)^2\right)\, ds.\end{aligned}$$

Interpretation of Nussbaum (1996)

$$Q_\delta := N\left(\frac{n}{2} + n\delta, \frac{n}{4}\right) \quad \text{and} \quad P_\delta := \text{Bin}(n, \tfrac{1}{2} + \delta)$$

Under quantile coupling, $X \sim P_0$ and $Y \sim Q_0$, and

$$Y \mid X = x \sim K_x := Q_0\left(\cdot \mid [\beta_{x,n}, \beta_{x+1,n})\right)$$

Define $\widetilde{Q}_\delta$ as the distribution of $\widetilde{Y}$, where

$$X \sim P_\delta \qquad \text{and} \qquad \widetilde{Y} \mid X = x \sim K_x$$

Hellinger distance:

$$H^2(\widetilde{Q}_\delta, Q_\delta) = O(\delta^2) \qquad \text{for } \delta = O(n^{-1/2}).$$

Carter (2000, 2001)

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