MINIMAX THEOREM

Sources: *Kneser? Sion? See Millar (1983, page 92). Possibly also some handwritten notes from other lectures by Millar.*

extend to $\mathbb{R} \cup \{\infty\}$ valued f?

<1> **Theorem.** *Let K be a compact convex subset of a Hausdorff topological vector space $\mathcal{X}$, and C be a convex subset of a vector space $\mathcal{Y}$. Let f be a real-valued function defined on $K \times C$ such that*

(i) $x \mapsto f(x,y)$ is convex and lower-semicontinuous for each y,

(ii) $y \mapsto f(x,y)$ is concave for each x.

Then

$$\inf_{x\in K} \sup_{y\in C} f(x,y) = \sup_{y\in C} \inf_{x\in K} f(x,y).$$

Proof. Note that assumption (i) means precisely that $K_{y,t} := \{x \in K : f(x,y) \le t\}$ is compact and convex, for each fixed $t \in \mathbb{R}$ and $y \in C$.

Clearly the left-hand side of the asserted equality is $\ge$ the right-hand side. It therefore suffices to prove that if M is a real number for which $M \ge$ right-hand side then, for each $\epsilon > 0$, the left-hand side is $\le M + \epsilon$. The assumption on M means that $\inf_{x\in K} f(x,y) \le M$, and hence $K_{y,M+\epsilon} \ne \emptyset$, for every y. If we can prove that $\cap_{y\in\mathcal{Y}} K_{y,M+\epsilon} \ne \emptyset$ then there will exist an x_0 for which $f(x_0,y) \le M + \epsilon$ for every y, implying that $\sup_y f(x_0,y) \le M + \epsilon$.

Replacing f by $f - (M+\epsilon)$, we reduce to the case where $M + \epsilon = 0$ and $\inf_{x\in K} f(x,y) \le -\epsilon$ and $K_{y,0} \ne \emptyset$, for each y. We need to show that $\cap_{y\in\mathcal{Y}} K_{y,0} \ne \emptyset$. Compactness of each $K_{y,0}$ simplifies the task to showing that $\cap_{y\in\mathcal{Y}_0} K_{y,0} \ne \emptyset$ for each finite subset $\mathcal{Y}_0$ of $\mathcal{Y}$. An inductive argument will then reduce even further to the case where $\mathcal{Y}_0 := \{y_1, y_2\}$, a two-point set, which is the case that I consider first.

Abbreviate $K_{y_i,0}$ to K_i, and $f(x,y_i)$ to $f_i(x)$. Thus $K_i = \{x : f_i(x) \le 0\}$. For the purposes of obtaining a contradiction, suppose $K_1 \cap K_2 = \emptyset$. The contradiction will appear if we find a number α in $[0,1]$ for which

<2> $$(1-\alpha) f_1(x) + \alpha f_2(x) \ge 0 \qquad \text{for all } x \text{ in } K,$$

for then the concavity of $f(x,\cdot)$ would give the lower bound $\inf_{x\in K} f(x,y_\alpha) \ge 0$, where $y_\alpha := (1-\alpha)y_1 + \alpha y_2$.

Inequality <2> is trivial if $x \notin K_1 \cup K_2$, for then both $f_1(x) > 0$ and $f_2(x) > 0$. For it to hold at each x in K_1 we would need

<3> $$\alpha \ge \sup_{x_1\in K_1} \frac{-f_1(x_1)}{f_2(x_1) - f_1(x_1)}.$$

Notice that the supremum on the right-hand side is ≥ 0. For inequality <2> to hold at each x in K_2 we would need

<4> $$\alpha \le \inf_{x_2\in K_2} \frac{f_1(x_2)}{f_1(x_2) - f_2(x_2)}.$$

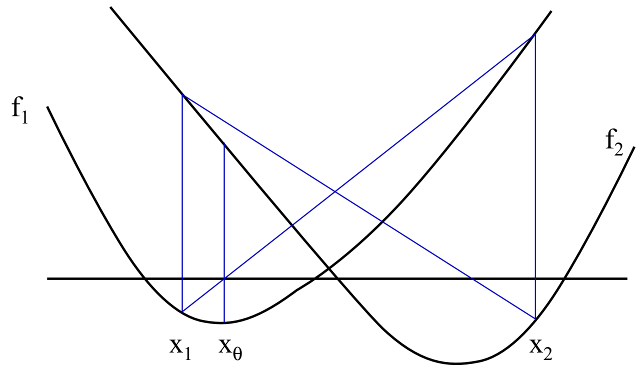
Notice that the infimum on the right-hand side is ≤ 1. There exists an α satisfying both constraints <3> and <4> if and only if

$$\frac{-f_1(x_1)}{f_2(x_1) - f_1(x_1)} \le \frac{f_1(x_2)}{f_1(x_2) - f_2(x_2)} \qquad \text{for all } x_1 \in K_1 \text{ and } x_2 \in K_2.$$

That is, α exists if and only if

<5> $$(-f_1(x_1))\,(-f_2(x_2)) \le f_1(x_2)\,f_2(x_1) \qquad \text{for all } x_1 \in K_1 \text{ and } x_2 \in K_2.$$

This inequality involves the values of the convex functions only along the line joining x_1 and x_2; it is essentially a one-dimensional result.



The inequality <5> is trivial when either $f_1(x_1) = 0$ or $f_2(x_2) = 0$. We need only consider a pair with $f_1(x_1) < 0$ and $f_2(x_2) < 0$. Define θ in $(0, 1)$ as the value for which $(1-\theta)\,f_1(x_1)+\theta f_1(x_2) = 0$, then define $x_\theta := (1-\theta)x_1+\theta x_2$. By convexity, $f_1(x_\theta) \le 0$, implying that $x_\theta \in K_1$ and $(1-\theta)\,f_2(x_1) + \theta f_2(x_2) \ge f_2(x_\theta) > 0$. Thus

$$\frac{-f_1(x_1)}{f_1(x_2)} = \frac{\theta}{1-\theta} < \frac{f_2(x_1)}{-f_2(x_2)},$$

which gives <5>.

Existence of an α satisfying constraints <3> and <4> now follows, which, via the contradiction, lets us conclude that $K_1 \cap K_2 \ne \emptyset$.

To extend the conclusion to an intersection of finitely many sets $K_{y_i,0}$, for $i = 1, 2, \dots, m$, first invoke the result for pairs to see that $K_i' := K_{y_1,0} \cap K_{y_i,0} \ne \emptyset$ for $i = 2, \dots, m$, then repeat the pairwise argument with f restricted to $K_{y_1,0} \times C$. And so on. After $m-1$ repetitions we reach the desired conclusion, that $\cap_{i\le m} K_{y_i,0} \ne \emptyset$. □

References

Millar, P. W. (1983), ‘The minimax principle in asymptotic statistical theory’, *Springer Lecture Notes in Mathematics* **976**, 75–265.