Maximum likelihood in an infinite-dimensional exponential family

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In honor of Jon Wellner's 65th birthday

Based on:
Dou, Pollard, and Zhou (2010) "Functional regression for general exponential families"
arXiv:1001.3742v1 [math.ST] 21 Jan 2010

Slides for this talk at: www.stat.yale.edu/~pollard/Talks

1. Problem: estimate unknown β

- Observe independent $(y_1, x_1), (y_2, x_2), \ldots$ with $y_i \mid x_i \sim Q_{\theta_i}$ and (nonrandom or condition) $x_i' = (x_{i1}, x_{i2}, \ldots)$ and $\theta_i = \langle x_i, \beta \rangle$
- $$\frac{dQ_\theta}{dQ_0} = \exp\left(y\theta - \Psi(\theta)\right) \qquad \theta \in \mathbb{R}$$
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Example: for $Q_\theta = \mathrm{Bernoulli}\left(\frac{e^\theta}{1+e^\theta}\right)$ on $\{0, 1\}$

$$\Psi(\theta) = \log\left(\frac{1+e^\theta}{2}\right)$$

Example: for $Q_\theta = \mathrm{Poisson}\left(e^\theta\right)$ on $\{0, 1, 2, \ldots\}$

$$\Psi(\theta) = e^\theta - 1$$

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- $$\frac{dQ_\theta}{dQ_0} = \exp\left(y\theta - \Psi(\theta)\right) \qquad \theta \in \mathbb{R}$$
- Ψ convex with $\Psi(0) = 0$
- MLE: maximize a concave function of b?

$$\widehat{b} \stackrel{?}{=} \operatorname*{argmax}_{b \in ?} \textstyle\sum_{i \leq n} \Big(y_i \langle x_i, b \rangle - \Psi(\langle x_i, b \rangle) \Big)$$

Component of $\widehat{b}$ in $\mathrm{span}\{x_1, \ldots, x_n\}^\perp$ is unconstrained.

2. Simplify: $x_i \in \mathbb{R}^N$, with $N = N_n$?

- MLE

$$\widehat{b}_n = \operatorname*{argmax}_{b \in \mathbb{R}^N} \sum_{i \leq n} \left(y_i x_i' b - \Psi(x_i' b) \right) \qquad \text{(concave)}$$

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- $\widehat{b}_n = \beta + J_n^{-1/2} \widehat{t}_n$ where $\widehat{t}_n = \underset{t \in \mathbb{R}^N}{\text{argmax}} L_n(t)$ for

$$L_n(t) = \sum_{i \le n} \left(y_i w_i' t - \Psi(\theta_i + w_i' t) \right) \qquad \text{(concave)}$$

3. Why reparametrize?

- [from last slide] $\widehat{t}_n = \operatorname*{argmax}_{t \in \mathbb{R}^N} L_n(t)$ where $L_n(t) = \sum_{i \le n} \Big(y_i w_i' t - \Psi(\theta_i + w_i' t) \Big)$

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\dot{L}(t) &= \frac{\partial}{\partial t} L_n(t) \\
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&= T_n' - t' \sum_{i \le n} w_i w_i' \ddot{\Psi}(\theta_i) - R_n(t)' \\
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- Should have $\widehat{t}_n \approx T_n$ (even when $N_n \to \infty$?)

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- Assume

$$|\dddot{\Psi}(\theta + h)| \leq \ddot{\Psi}(\theta) G(|h|) \qquad \text{for all } \theta \text{ and } h$$

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$$|R_n(t)| = |\sum_{i \leq n} \left(\dot{\Psi}(\theta_i + w_i' t) - \dot{\Psi}(\theta_i) - w_i' t \ddot{\Psi}(\theta_i) \right) w_i'|$$
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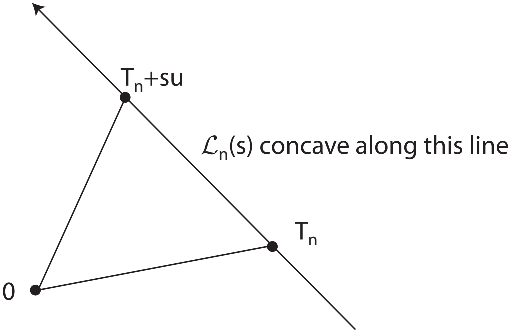

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$$\le \frac{\epsilon^2}{4G(1)N} G(1) \left(\frac{N}{\epsilon} + \epsilon^2 \right) < \frac{\epsilon}{2}$$

if

$$M_n \le \frac{\epsilon^2}{2G(1)N} \qquad \text{and} \qquad |T_n|^2 \le N/\epsilon$$

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- $\mathbb{P}\{|T_n|^2 > N/\epsilon\} < \epsilon$ because $\mathbb{P}|T_n|^2 = N$

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and

$$|J_n^{1/2}(\widehat{b}_n - \beta) - T_n| = |\widehat{t}_n - T_n| \le \epsilon$$

7. General case: $\theta_i = \langle x_i, \beta \rangle$ with $x_i \in \mathbb{R}^{\mathbb{N}}$

- Truncate $z = (z_1, z_2, \dots) = \pi_N z + \pi_N^{\perp} z$:

$$\pi_N z = (z_1, \dots, z_N, 0, 0, \dots), \qquad \pi_N^{\perp} z = (0, \dots, 0, z_{N+1}, z_{N+2}, \dots)$$

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  $$\mathbb{P}_n = \otimes_{i \leq n} Q_{\theta_i}, \qquad \mathbb{Q}_{n,N} = \otimes_{i \leq n} Q_{\theta_i^*}$$

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$$\begin{aligned} H^2(\mathbb{P}_n, \mathbb{Q}_{n,N}) &= \textstyle\sum_{i \le n} H^2(Q_{\theta_i}, Q_{\theta_i^*}) \\ &\le \textstyle\sum_{i \le n} \delta_i^2 \ddot{\Psi}(\theta_i)\,(1 + |\delta_i|)\, G(|\delta_i|) \end{aligned}$$

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- Assumptions on N_n and on decay of β_j's needed to make $\|\mathbb{P}_n - \mathbb{Q}_{n,N}\|_{\text{TV}} \to 0$ and $|\pi_N^{\perp}\beta|$ suitably small.
- Deduce $\mathbb{P}_n\{\widehat{b}_{n,N} \approx \pi_N \beta\} \approx 1$ if ....

References

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