Measures and integral representations

The following explanation streamlines, expands, and corrects the version of the arguments presented by Pollard (2001, Appendix A).

1 Inner measures

S:inner

The construction of a countably additive measure on a sigma-field on a set $\mathcal{X}$ usually starts from a set function μ defined on $\mathcal{K}$, a small collection of subsets of $\mathcal{X}$. One requires μ to have measure-like properties on $\mathcal{K}$ then seeks to extend μ to a larger domain.

The classical approach (Folland, 1999, Chapter 1) takes $\mathcal{K}$ as a field or a ring of sets. One defines an ***outer measure*** for every subset of $\mathcal{X}$ by

$$\mu^* A = \inf\{\sum\nolimits_i \mu K_i : A \subseteq \cup_{i\in\mathbb{N}} K_i, \quad K_i \in \mathcal{K}\}.$$

One then shows that

$$\mathcal{S}^* = \{S \subseteq \mathcal{X} : \mu^* A = \mu^*(AS) + \mu^*(AS^c) \text{ for every } A \subseteq \mathcal{X}\}$$

is a sigma-field and that the restriction $\widetilde{\mu}$ of μ^* to $\mathcal{S}^*$ is a countably additive measure. I believe this variation on Lebesgue's original argument is due to Carathéodory.

For many probabilistic purposes it is important to be able to approximate Borel sets from inside by compact sets, $\widetilde{\mu}B = \sup\{\widetilde{\mu}K : B \supseteq K, \quad K \text{ compact}\}$. Measures with this property are known as ***Radon measures***.

Topsøe (1970) took this desirable property as the starting point for the construction of measures. He took $\mathcal{K}$ to be an abstract set of subsets of $\mathcal{X}$ with properties analogous to those of the collection of all compact subsets of a Hausdorff topological space (such as $\mathbb{R}^d$). For a map $\mu : \mathcal{K} \to \mathbb{R}^+$ he adapted the Carathéodory approach for use with an ***inner measure***, defined by

inner.def <1>
$$\mu_* A = \sup\{\mu K : A \supseteq K \in \mathcal{K}\},$$

The role of the closed sets was taken over by

$$\mathcal{F}(\mathcal{K}) := \{F \subseteq \mathcal{X} : FK \in \mathcal{K} \text{ for every } K \in \mathcal{K}, \}$$

with $\mathcal{B}(\mathcal{K}) := \sigma(\mathcal{F}(\mathcal{K}))$ playing the role of the Borel sigma-field.

By analogy with the outer method construction, Topsøe identified

$$\mathcal{S}_* := \{S \subseteq \mathcal{X} : \mu_* A = \mu_*(AS) + \mu_*(AS^c) \text{ for every } A \subseteq \mathcal{X}\}$$

as a suitable domain for the extension.

In general, the set $\mathcal{K}$ is assumed to be a $(\emptyset, \cup f, \cap c)$ paving on $\mathcal{X}$, meaning that $\emptyset \in \mathcal{K}$ and that $\mathcal{K}$ is stable under finite unions and countable intersections.

Typically $\mathcal{K}$ is not stable under complements or differences, which makes it tricky to capture the idea that μ should at least have some sort of finite-additivity property. In place of finite-additivity Topsøe required that μ be $\mathcal{K}$-***tight***:

Note: $A \backslash B := AB^c := A \cap B^c$.

kk.tight <2>
$$\mu K_1 = \mu K_2 + \mu_*(K_1 \backslash K_2) \qquad \text{for all } K_1, K_2 \in \mathcal{K} \text{ with } K_1 \supseteq K_2.$$

As a surrogate for countable additivity of μ, Topsøe required that μ be ***sigma-smooth at*** $\emptyset$:

sigma.smooth.set <3>
$$\mu K_n \downarrow 0 \qquad \text{if } \{K_n : n \in \mathbb{N}\} \subset \mathcal{K} \text{ and } K_n \downarrow \emptyset.$$

Under these conditions the set $\mathcal{S}_*$ is a sigma-field and the restriction of μ_* to $\mathcal{S}_*$ is a countably additive measure, with the useful inner regularity property.

inner.reg <4> **Theorem.** *Suppose $\mathcal{K}$ is an $(\emptyset, \cup f, \cap c)$ paving on a set $\mathcal{X}$ and $\mu : \mathcal{K} \to \mathbb{R}^+$ is a $\mathcal{K}$-tight map that is sigma-smooth at $\emptyset$. Then there exists an extension of μ to a countably additive, $\mathcal{K}$-inner regular measure $\widetilde{\mu}$ on a sigma-field $\mathcal{S}_*$, which contains the sigma-field $\mathcal{B}(\mathcal{K})$ generated by $\mathcal{F}(\mathcal{K})$.*

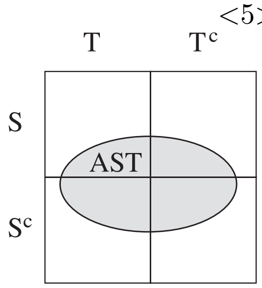
The term $\mathcal{K}$***-inner-regular*** means that $\widetilde{\mu}A = \sup\{\mu K : A \supseteq K \in \mathcal{K}\}$ for each A in $\mathcal{S}_*$. For the construction described in the remainder of the Section, $\mathcal{K}$-inner-regularity comes from the fact $\widetilde{\mu}(A) = \mu_*(A)$.

> **Remark.** Compare with Theorem 1 from Section IX.§3.1 of Bourbaki (2004b). Essentially their Lemma 1 established $\mathcal{K}$-tightness for the paving of all compact subsets of a Hausdorff topological space. Their result assumed a property stronger than σ-smoothness. See Section 5.

It helps to break the construction of the measure into two steps. First one shows that $\mathcal{S}_*$ is a field (stable under finite unions and intersections and complements) that contains $\mathcal{F}(\mathcal{K})$ and that the restriction of μ_* to $\mathcal{S}_*$ is finitely additive. Those properties simplify the second step, which translates the σ-smoothness into countable additivity.

1.1 Finite additivity

In fact the first part has very little to do with inner measures; the outer measure construction starts in the same way.



<5> **Task:** Show that $\mathcal{S}_*$ is stable under complements and pairwise intersections, that is, $\mathcal{S}_*$ is a field of subsets of $\mathcal{X}$. Show also that the restriction $\widetilde{\mu}$ of μ_* to $\mathcal{S}_*$ is a finitely additive measure. Argue as follows.

(i) If $S, T \in \mathcal{S}_*$ and $A \subseteq \mathcal{X}$ define $B = A \cap (ST)^c$. Show that

$$\begin{aligned} \mu_* A &= \mu_*(AST) + \mu_*(AS^cT) + \mu_*(AST^c) + \mu_*(AS^cT^c) \\ \mu_* B &= \phantom{\mu_*(AST) +{}} \mu_*(AS^cT) + \mu_*(AST^c) + \mu_*(AS^cT^c). \end{aligned}$$

(ii) If $S, T \in \mathcal{S}_*$ and $ST = \emptyset$ consider $A = S \cup T$.

$\square$

The $\mathcal{K}$-tightness property establishes the connection between $\mathcal{S}_*$ and $\mathcal{F}(\mathcal{K})$.

<6> **Task:** If μ is $\mathcal{K}$-tight show:

(i) μ is non-decreasing on $\mathcal{K}$, that is, $\mu K_1 \geq \mu K_2$ if $K_1 \supseteq K_2$

(ii) $\mu_*(K) = \mu K$ for each $K \in \mathcal{K}$

(iii) $\mu(K_1 \cup K_2) = \mu K_1 + \mu K_2$ if K_1 and K_2 are disjoint members of $\mathcal{K}$

(iv) $\mu\emptyset = 0$

(v) $\mu_* A \geq \mu_*(AB) + \mu_*(AB^c)$ for all subsets A, B of $\mathcal{X}$.

(vi) $S \in \mathcal{S}_*$ iff $\mu K \leq \mu_*(KS) + \mu_*(KS^c)$ for every K in $\mathcal{K}$

(vii) If $F \in \mathcal{F}(\mathcal{K})$ and $K \in \mathcal{K}$ then $\mu K = \mu(KF) + \mu_*(KF^c)$. Hint: $K_0 := KF \in \mathcal{K}$ and $KF^c = K\backslash K_0$. Deduce that $\mathcal{F}(\mathcal{K}) \subseteq \mathcal{S}_*$.

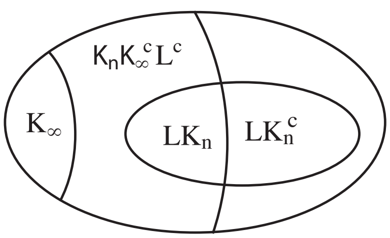
□

Now we know that all elements of $\mathcal{F}(\mathcal{K})$, which includes $\mathcal{K}$, belong to $\mathcal{S}_*$. For calculations involving only sets in $\mathcal{S}_*$ we can use the finite additivity of μ_*, which makes some of the approximation arguments more familiar for anyone who is used to working with the linearity properties of integrals.

1.2 Countable additivity

First extend the σ-smoothness to decreasing sequences of $\mathcal{K}$-sets whose intersection (which necessarily belongs to $\mathcal{K}$) might not be empty.

<7> **Task:** Suppose μ is a map from $\mathcal{K}$ to $\mathbb{R}^+$ that is $\mathcal{K}$-tight and sigma-smooth at $\emptyset$. If if $\{K_n : n \in \mathbb{N}\} \subset \mathcal{K}$ and $K_n \downarrow K_\infty$, show that $\mu K_n \downarrow \mu K_\infty$. Argue as follows. Given $\epsilon > 0$ choose $L \in \mathcal{K}$ with $L \subseteq K_1\backslash K_\infty$ and $\epsilon + \mu L > \mu_*(K_1\backslash K_\infty)$. Then explain why



$$\mu K_n \leq \mu K_\infty + \mu(K_n L) + \mu_*(K_1 K_\infty^c L^c).$$

Note that $K_n L \downarrow \emptyset$. Use the finite additivity.

□

With $\mathcal{K}$-tightness and sigma-smoothness you have enough to prove that $\mathcal{S}_*$ is a sigma-field and that μ_* defines a countably additive measure on $\mathcal{S}_*$. For sequences $\{A_i : i \in \mathbb{N}\}$ in $\mathcal{S}_*$, you need to show:

(i) $S := \cup_{i\in\mathbb{N}} A_i$ belongs to $\mathcal{S}_*$

(ii) If the A_i's are disjoint then $\mu_* S = \sum_{i\in\mathbb{N}} \mu_* A_i$.

This task is made easier because we already know that $\mathcal{S}_*$ is a field and that μ_* is finitely additive on $\mathcal{S}_*$. It is easier to work with the sets $S_n = \cup_{i \le n} A_i$, for which $S_n \uparrow S$ and, if the A_i's are disjoint, $\mu_* S_n = \sum_{i \le n} \mu_* A_i$.

<8> **Task:** Suppose $\{S_n : n \in \mathbb{N}\}$ in $\mathcal{S}_*$ and $S_n \uparrow S$. Show that $S \in \mathcal{S}_*$ and $\mu_* S_n \uparrow \mu_* S$ by the following steps.

(i) Consider an arbitrary K in $\mathcal{K}$. Show that $KS_n^c \downarrow KS^c$ and

$$\mu K \le \mu_*(KS) + \mu_*(KS_n^c).$$

(ii) Prove that $\mu_*(KS_n^c) \downarrow \mu_*(KS^c)$. Argue as follows. Given $\epsilon > 0$ choose $L_i \in \mathcal{K}$ with $L_i \subseteq KS_i^c$ and $\mu_*(KS_i^c L_i^c) < \epsilon/2^i$. Define $K_n = \cap_{i \le n} L_i$ and $K_\infty = \cap_{i \in \mathbb{N}} L_i$.

(a) Show that $K_n \subseteq KS_n^c$ and $K_\infty \subseteq KS^c$ and $\mu_*(KS_n^c K_n^c) < \epsilon$.

(b) Use sigma-smoothness at K_∞ to deduce that

$$\lim_n \mu_*(KS_n^c) \le \epsilon + \mu K_\infty \le \epsilon + \mu_*(KS^c).$$

(iii) Deduce that $\mu K \le \mu_*(KS) + \mu_*(KS^c)$ and $S \in \mathcal{S}_*$.

□

It remains only to show that the restriction of μ_* to $\mathcal{S}_*$ is countably additive. The notation continues from Task 4.

<9> **Task:**

We know that $\mu_* S \ge \mu_* S_n \uparrow C$ for some (possibly infinite) C. To avoid some messy details with the case $\mu_* S = \infty$ consider any t for which $\mu_* S > t \in \mathbb{R}^+$. We have only to prove that $C \ge t$.

Choose $K \subseteq S$ with $\mu K > t$. Show that

$$t < \mu K = \mu_*(KS_n) + \mu_*(KS_n^c) \le C + \mu_*(KS_n^c).$$

Use (ii) from Task 4 to dispose of $\lim_{n\to\infty} \mu_*(KS_n)$.

□

Now you should put together all the pieces to establish Theorem <4>.

2 Lebesgue measure on Euclidean space

S:Lebesgue

Let $\mathcal{K}$ denote the set of all compact subsets of $\mathbb{R}^k$ and let $\mathcal{I}$ denote the set of all open rectangles of the form $I = (a_1, b_1) \times \cdots \times (a_k, b_k)$ and $\overline{\mathcal{I}}$ denote the set of all closed rectangles of the form $\overline{I} = [a_1, b_1] \times \cdots \times [a_k, b_k]$ Define $v(I) = v(\overline{I}) = \prod_i (b_i - a_i)$ for their volumes.

For each K in $\mathcal{K}$ define

Leb <10> $$\mu K = \inf\{\sum_i v(I_i) : K \subset \cup_i I_i\}$$

where the infimum runs over all all finite coverings of K by intervals from $\mathcal{I}$.

<11> **Task:** Show that $\mu K = \prod_i (b_i - a_i)$ if $K = \prod_i [a_i, b_i]$. The inequality $\mu K \le \prod_i (b_i - a_i + 2\epsilon)$ for each $\epsilon > 0$ is easy.

Here is a suggestion for the reverse inequality. For simplicity suppose $k = 2$ and $K = [a_0, b_0] \times [c_0, d_0] \subseteq J_1 \cup \ldots J_n$, where $J_\alpha = (a_\alpha, b_\alpha) \times (c_\alpha, d_\alpha)$. There exist real numbers $s_0 < s_1 < \cdots < s_\ell$ and $t_0 < t_1 < \cdots < t_m$ for which K and each $\overline{J}_\alpha$ can be decomposed into a union of intervals of the form $\overline{I}_{i,j} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$.

□

Notice that, for each $\delta > 0$, we may assume that each I_i in <10> has diameter at most δ. For example, for arbitrarily large n the rectangle $I = (0,1) \times (0,1)$ is covered by $\bigcup_{i,j} \overline{I}_{i,j}$ where

$$\overline{I}_{i,j} = \{(x,y) : i/n \le x \le (i+1)/n, \quad j/n \le y \le (j+1)/n\}$$

and $v(I) = \sum_{I,j} v(\overline{I}_{i,j})$. Then each of those closed rectangles can be expanded slightly to an open rectangle.

subadd <12> **Task:** For all $K, L \in \mathcal{K}$ show that $\mu(K \cup L) \le \mu K + \mu L$, with equality if $K \cap L = \emptyset$. For the second part note that if the distance between K and L is greater than δ and if each I_i in a covering $\cup_i I_i$ of $K \cup L$ has diameter less than δ then no I_i can intersect both K and L.

□

The proof of sigma-smoothness is made trivial by the compactness. If $K_i \in \mathcal{K}$ and $K_1 \supseteq K_2 \supseteq \ldots \downarrow \emptyset$ then $K_1 \subset \cup_{i \ge 2} K_i^c$. By compactness, $K_1 \subseteq K_m^c$ for some m, which forces $K_m = \emptyset$.

The other requirement of Theorem <4> is only slightly more difficult to establish.

<13> **Task:** Prove that μ is $\mathcal{K}$-tight.

Suppose $K_1, K_2 \in \mathcal{K}$ and $K_1 \supset K_2$. Use Task 12 to show that $\mu K_1 \geq \mu K_2 + \mu_*(K_1 \backslash K_2)$.

For the reverse inequality, cover K_2 by $G = \cup_i I_i$ with $\epsilon + \mu K_2 > \sum_i v(I_i)$. Note that $L := K_1 \backslash G$ is a subset of $K_1 \backslash K_2$. Find a cover $\cup_j J_j$ of L for which $\epsilon + \mu L > \sum_j v(J_j)$. Show that the finite set of I_i and J_j rectangles cover K_1 so that $\mu K_1 \leq \sum_i v(I_i) + \sum_j v(J_j) \leq 2\epsilon + \mu K_2 + \mu L$.

□

3 The Kolmogorov extension theorem

S:Kolmogorov

4 Linear functionals as integrals

S:Daniell

Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$. Chapter 2 of Pollard (2001) exploited the natural correspondence between measures μ on $\mathcal{A}$ and increasing 'linear' functionals (integrals) on the cone $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ of all measurable functions taking values in $[0, \infty]$.

lin.fnal <14> **Theorem.** *For each measure μ on $(\mathcal{X}, \mathcal{A})$ there is a uniquely determined functional, a map T from $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ into $[0, \infty]$, having the following properties:*

(i) $T(\mathbb{1}_A) = \mu A$ for each A in $\mathcal{A}$;

(ii) $T(0) = 0$, where the first zero stands for the zero function;

'LINEAR' *(iii) for nonnegative real numbers α, β and functions f, g in $\mathcal{M}^+$,*

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g);$$

INCREASING *(iv) if f, g are in $\mathcal{M}^+$ and $f \leq g$ everywhere then $T(f) \leq T(g)$;*

MONOTONE CONVERGENCE PROPERTY *(v) if $f_1, f_2, \dots$ is a sequence in $\mathcal{M}^+$ with $0 \leq f_1(x) \leq f_2(x) \leq \dots \uparrow f(x)$ for each x in $\mathcal{X}$ then $T(f_n) \uparrow T(f)$.*

The monotone convergence property for the integral is an analog of the countable additivity of the measure.

Conversely, if T is a map from $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ into $[0, \infty]$ that has properties (ii)—(v) then there is countably additive measure μ on $\mathcal{A}$ defined by $\mu(A) = T(\mathbb{1}_A)$.

The term 'linear' might be misleading because the domain $\mathcal{M}^+$ is not a vector space. The true linearity emerges after the integral is extended to functions taking both positive and negative real values. When restricted to the vector space $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$, the integral is a real-valued linear functional, with the countable additivity of the measure implying other results (such as Dominated Convergence) that allow interchange in the order of pointwise limits and integration. We gain flexibility by making the domain a vector space but at the cost of having to be more careful about possibly infinite limits or about contributions froim sets with small measure.

The construction of the integral involves little work once we have the measure on a sigma-field; and recovery of the measure from the functional, via $\mu A = T(\mathbb{1}_A)$, is a triviality because $\mathbb{1}_A$ belongs to the domain of T.

Another approach to integration (due to Daniell, 1918—see Segal and Kunze, 1978, Section 3.3) starts with a linear functional T on a 'small' vector space $\mathcal{H}$ of real-valued functions and seeks to extend that functional to a much larger collection of functions. Only later does one try to recover the underlying measure and interpret the extended functional as an integral with respect to that measure. For example, for Lebesgue measure on $[0, 1]$ we might start with the functional $Tf = \int_0^1 f(x)\,dx$ defined for the set of all continuous real functions on $[0, 1]$ by some sort of Riemann integral. The Daniell construction then extends the domain of T to include at least all bounded, Borel-measurable functions. One can then recover the Lebesgue measure of Borel sets via the extended functional, which can be interpreted as an integral with respect to Lebesgue measure.

Topsøe (1970, Section 3) carried out a Daniell-like extension procedure using an "inner integral" approach. Pollard and Topsøe (1975) modified that approach by using the functional T on $\mathcal{H}$ to construct a $\mathcal{K}$-tight set function on a $(\emptyset, \cup f, \cap c)$ paving $\mathcal{K}$ defined by $\mathcal{H}$. The method described in Section 1 provides the extension of the measure to a suitably large sigma-field. The integral with respect to that measure defines the extension/representation for the functional. This approach has the advantage of recovering a large variety of integral representation theorems in one fell swoop.

The simplest case occurs when $\mathcal{H}$ is a vector lattice of real-valued functions, that is, a vector space that is stable under the operations of pointwise maximum or minimum of pairs of functions:

$$(h_1 \vee h_2)(x) := \max(h_1(x), h_2(x)) \quad \text{AND} \quad (h_1 \wedge h_2)(x) := \min(h_1(x), h_2(x)).$$

It is not necessary to assume that the constant functions belong to $\mathcal{H}$. Instead one assumes that $\mathcal{H}$ satisfies Stone's condition

Stone <15> $$1 \wedge h \in \mathcal{H} \qquad \text{for each } h \in \mathcal{H}.$$

Here it might be better to write $\mathbb{1} \wedge h$ to emphasize that the 1 represents the constant function taking value 1, which need not belong to $\mathcal{H}$.

> **Remark.** Think of the case where $\mathcal{H}$ equals the set of all continuous functions with compact support on the real line. The function $\mathbb{1}$ is continuous but the set $\{x \in \mathbb{R} : \mathbb{1}(x) \neq 0\}$ does not have compact closure.

Daniell <16> **Theorem.** *Let $\mathcal{H}$ be a vector lattice of real-valued functions satisfying Stone's condition, defined on a set $\mathcal{X}$. Let $T : \mathcal{H} \to \mathbb{R}$ be an increasing linear functional for which*

sigma.smooth.fnal <17>
$$T(h_n) \downarrow 0 \qquad \text{if } \{h_n : n \in \mathbb{N}\} \subset \mathcal{H} \text{ and } h_n(x) \downarrow 0 \text{ for each } x.$$

Then there exists an $(\emptyset, \cup f, \cap c)$ paving $\mathcal{K}$ on $\mathcal{X}$ and a $\mathcal{K}$-tight function μ on $\mathcal{K}$ that is sigma-smooth at $\emptyset$ for which:

(i) Each h in $\mathcal{H}$ is $\mathcal{B}(\mathcal{K})$-measurable.

(ii) The $\mathcal{K}$-inner-regular extension $\widetilde{\mu}$ of μ is the unique $\mathcal{K}$-inner-regular measure for which $T(h) = \widetilde{\mu}h$ for each $h \in \mathcal{H}$.

> **Remark.** In fact $\mathcal{B}(\mathcal{K})$ will be the sigma-field generated by $\mathcal{H}$, the smallest sigma-field $\sigma(\mathcal{H})$ for which each member of $\mathcal{H}$ is $\sigma(\mathcal{H})\backslash\mathcal{B}(\mathbb{R})$-measurable. If $\mathcal{H}$ were the set of all bounded continuous real functions on $\mathcal{X}$, this $\sigma(\mathcal{H})$ is often called the ***Baire sigma-field***.
>
> For separable metric spaces the Baire and Borel sigma-fields are the same.
>
> If $\mathcal{H}$ were the set of all continuous real functions with compact support on a locally compact Hausdorff space then $\sigma(\mathcal{H})$ could be smaller than the Borel sigma-field.
>
> See Section 5 for a condition stronger than σ-smoothness that leads to measures on larger sigma-fields.

Assumption <17> is sometimes called ***sigma-smoothness at zero***, an analogue of <3>. Because $\mathcal{H}$ is a vector space and T takes real values, <17> is equivalent to either of

$$T(h_n) \downarrow T(h_\infty) \text{ if } \{h_n : n \in \mathbb{N} \cup \{\infty\}\} \subset \mathcal{H} \text{ and } h_n(x) \downarrow h_\infty(x) \text{ for each } x$$
$$T(h_n) \uparrow T(h_\infty) \text{ if } \{h_n : n \in \mathbb{N} \cup \{\infty\}\} \subset \mathcal{H} \text{ and } h_n(x) \uparrow h_\infty(x) \text{ for each } x.$$

The equivalence comes from replacement of h_n by $h_n - h_\infty$ or $h_\infty - h_n$. Note well that the equivalence depends on the assumption that the limit function h_∞ belongs to $\mathcal{H}$.

Once again I'll break the proof of the Theorem into a sequence of tasks.

<18> **Task:** Define $\mathcal{K}$ as the set of all subsets K of $\mathcal{X}$ whose indicator functions are pointwise infima of countable sets of $\mathcal{H}$ functions: $\mathbb{1}_K = \inf \mathcal{H}_0$ where $\mathcal{H} \subset \mathcal{H}$ and $\mathcal{H}_0$ is countable. Equivalently, if $\mathcal{H}_0 = \{h_i : i \in \mathbb{N}\}$ we could replace h_n by $\min(h_1, \ldots, h_n)$ so that $h_n(x) \downarrow \mathbb{1}\{x \in K\}$.

(i) Show that $\mathcal{K}$ is a $(\emptyset, \cup f, \cap c)$ paving on $\mathcal{X}$.

(ii) Show that $t \wedge h$ and $(h - t)^+$ are in $\mathcal{H}$ if $h \in \mathcal{H}$ and $t > 0$ is a constant.

(iii) If $h \in \mathcal{H}$ and $t > 0$ is a constant show that

$$\inf_{n > 1/t} \left[1 \wedge ((n+1)h(x) - (nt - 1))\right]_+ = \mathbb{1}\{x : h(x) \geq t\}.$$

Deduce that $\{h \geq t\} \in \mathcal{K}$.

□

<19> **Task:** Continue the notation from the previous Task. Now show that $\mathcal{B}(\mathcal{K})$ is big enough for integrals of $\mathcal{H}$-functions to be well defined.

(i) If $K \in \mathcal{K}$ with $\mathbb{1}_K = \inf\{h_n : n \in \mathbb{N}\}$ and $t > 0$, show that

$$\inf_{n \in \mathbb{N}} \left(h_n(x) - n\left(h(x) - t\right)_+\right)_+ = \mathbb{1}\{x \in K : h(x) \leq t\}.$$

Deduce that $K\{h \leq t\} \in \mathcal{K}$ and $\{h \leq t\} \in \mathcal{F}(\mathcal{K})$.

(ii) If $h \in \mathcal{H}$ and $1 \geq h \geq 0$ show that

$$h(x) = \lim_{k \to \infty} k^{-1} \sum_{j=1}^{k} \mathbb{1}\{h(x) \geq j/k\}.$$

Deduce that h is $\mathcal{B}(\mathcal{K})$-measurable.

(iii) By splitting into positive and negative parts and taking limits of suitably truncated functions, show that every h in $\mathcal{H}$ is $\mathcal{B}(\mathcal{K})$-measurable.

□

I could define $\mu(K)$ for $K \in \mathcal{K}$ by taking limits of $T(h_n)$ for sequences $h_n \downarrow \mathbb{1}_K$ but it seems cleaner to define μ in a way that shows it does not depend on the choice of the decreasing sequence $\{h_n\}$.

First some preliminaries showing how linearity of T implies some additivity properties for μ.

<20> **Task:** For $K \in \mathcal{K}$ define $\mu K := \inf\{T(h) : \mathbb{1}_K \leq h \in \mathcal{H}\}$.

(i) If $h_n \downarrow \mathbb{1}_K$ and $h \geq \mathbb{1}_K$ show that $h_n \vee h \downarrow h$. Deduce that

$$T(h_n) \leq T(h_n \vee h) \downarrow T(h).$$

Conclude that $T(h_n) \downarrow \mu K$.

(ii) Suppose K_2 and L are disjoint $\mathcal{K}$-sets and $\{f_n\}$ and $\{g_n\}$ are sequences of $\mathcal{H}$ functions for which $f_n \downarrow \mathbb{1}_{K_2}$ and $g_n \downarrow \mathbb{1}_L$. Show that $f_n \vee g_n \downarrow \mathbb{1}_{K_2 \cup L}$ and $f_n \wedge g_n \downarrow 0$. From the equality

$$T(f_n) + T(g_n) = T(f_n \vee g_n) + T(f_n \wedge g_n)$$

deduce that $\mu(K_2) + \mu(L) = \mu(K_2 \cup L)$.

□

Now we get to the heart of the proof, the $\mathcal{K}$-tightness and σ-smoothness.

<21> **Task:**

(i) For $\mathcal{K}$-sets K_1, K_2 with $K_1 \supseteq K_2$ deduce that $\mu(K_1) \geq \mu(K_2) + \mu_*(K_1 \backslash K_2)$.

(ii) Suppose $K_1, K_2 \in \mathcal{K}$ and $K_1 \supseteq K_2$. Argue as follows to show that $\mu(K_1) \leq \mu(K_2) + \mu_*(K_1 \backslash K_2)$, thus proving that μ is $\mathcal{K}$-tight.

(a) Suppose $\mathbb{1}_{K_2} \leq g \in \mathcal{H}$ and $t \in (0,1)$. Define $L = K_1\{g \leq t\}$. Show that $L \in \mathcal{K}$ and $L \subseteq K_1 \backslash K_2$. Thus $\mu L \leq \mu_*(K_1 \backslash K_2)$.

(b) Suppose $\mathbb{1}_L \leq h \in \mathcal{H}$. Show that $h + (g/t) \geq \mathbb{1}_{K_1}$. Deduce that $T(h) + t^{-1}T(g) \geq \mu K_1$.

(c) Take an infimum over h then over g, then let t increase to 1.

□

<22> **Task:**

Prove that μ is sigma-smooth at $\emptyset$. Argue as follows. Suppose $K_n \downarrow \emptyset$ and $h_{i,j} \downarrow \mathbb{1}_{K_i}$ as $j \to \infty$, for each fixed i. Show that $h_n := \min_{i,j \leq n} h_{i,j} \geq \mathbb{1}_{K_n}$

and $h_n \downarrow 0$ as $n \to \infty$. (If $x \in K_n^c$ then $h_{i,j}(x) \downarrow 0$ if $i \geq n$.) Deduce that $\mu K_n \leq T(h_n) \to 0$.]

□

Theorem <4> now guarantees that μ extends to a $\mathcal{K}$-inner-regular measure $\widetilde{\mu}$ on a sigma-field $\mathcal{S}_*$ that is larger than $\mathcal{B}(\mathcal{K})$. Each function in $\mathcal{H}$ is $\mathcal{B}(\mathcal{K})$-measurable.

To prove that each h in $\mathcal{H}$ is $\widetilde{\mu}$-integrable and that $T(h) = \widetilde{\mu}(h)$ it suffices to consider the case where $h \geq 0$ (split into $h^+ - h^-$) and to assume h is bounded (cf. $h \wedge n \uparrow h$). Thus we may assume $1 \geq h \geq 0$.

The argument uses an approximation by simple functions. Let k be a positive integer. Define f_k by rounding h down to an integer multiple of k^{-1}. That is,

$$f_k(x) = k^{-1} \sum\nolimits_{j=1}^{k} \mathbb{1}\{x \in L_j\} \qquad \text{where } L_j = \{h \geq j/k\}.$$

You might remember that $f_k(x) \leq h(x) \leq f_k(x) + k^{-1}$ for every x. We need to sharpen the upper bound slightly, replacing the k^{-1} by a function in $\mathcal{H}$. You should check that

$$f_k(x) \leq h(x) \leq h(x) \leq f_k(x) + \big(h(x) \wedge k^{-1}\big) \qquad \text{for each } x.$$

If $k = 2^m$, then f_k increases monotonely to h as m increases. Here are the details.

<23> **Task:** For fixed k find $\mathcal{H}$ functions $h_{j,n}$ for which $h_{j,n}(x) \downarrow \mathbb{1}\{x \in L_j\}$ as $n \to \infty$. Define $h_n(x) := k^{-1} \sum_{j=1}^{k} h_{j,n}(x)$.

(i) Show that $h_n(x) \downarrow f_k(x)$ as $n \to \infty$ and

$$T(h_n) \downarrow k^{-1} \sum\nolimits_{j=1}^{k} \mu L_j = \widetilde{\mu}(f_k).$$

(ii) Show that $h \vee h_n \downarrow h$. Deduce that

$$T(h) = \lim\nolimits_n T(h \vee h_n) \geq \lim\nolimits_n T(h_n) = \widetilde{\mu}(f_k).$$

Pass to the limit as $k \to \infty$ to conclude that $T(h) \geq \widetilde{\mu}(h)$.

(iii) Show that $(h \wedge k^{-1}) + h_n \geq h$. Apply T to both sides of the inequality, let n tend to infinity, then let k tend to ∞ to deduce that $\widetilde{\mu}h \geq T(h)$.

(iv) (uniqueness) Suppose ν is another $\mathcal{K}$-inner-regular measure for which $T(h) = \nu(h)$ for each h in $\mathcal{H}$. If $K \in \mathcal{K}$ and $h_n \downarrow \mathbb{1}_K$ show that

$$\nu K = \lim_n \nu(h_n) = \lim_n T(h_n) = \mu(K).$$

The measures ν and μ must agree for every measurable set B for which $\nu B = \sup\{\nu K : B \supseteq K \in \mathcal{K}\}$ and $\mu B = \sup\{\mu K : B \supseteq K \in \mathcal{K}\}$.

□

5 Extension to a larger sigma-field

S:tau

6 Measures on locally compact spaces

S:loc.cp

Bourbaki (2004a, page INT III.7) defined a measure (possibly signed or complex-valued) on a locally compact space to be a continuous linear functional on the vector space of continuous (complex-valued) functions with compact support. That is, for Bourbaki the integral as a linear functional was the primary concept and the representation as an integral with respect to a countably additive measure was secondary.

References

Bourbaki_integration_I

Bourbaki, N. (2004a). *Integration I: Chapters 1–6.* Elements of Mathematics. Springer-Verlag. English translation by S. K. Berberian of *Intégration* 1965, 1967, 1959.

Bourbaki_integration_II

Bourbaki, N. (2004b). *Integration II: Chapters 7–9.* Elements of Mathematics. Springer-Verlag. English translation by S. K. Berberian of *Intégration* 1965, 1967, 1959.

Daniell18integral

Daniell, P. J. (1918). A general form of integral. *Annals of Mathematics (series 2) 19*, 279–294.

Folland99

Folland, G. B. (1999). *Real Analysis: Modern Techniques and Their Applications.* Wiley.

PollardUGMTP

Pollard, D. (2001). *A User's Guide to Measure Theoretic Probability.* Cambridge University Press.

PollardTopsoe:75Studia

Pollard, D. and F. Topsøe (1975). A unified approach to Riesz type representation theorems. *Studia Mathematica 54*, 173–190.

SegalKunze1978

Segal, I. E. and R. A. Kunze (1978). *Integrals and Operators* (Second ed.). Number 228 in Grundlehren der mathematischen Wissenschaften. Springer Verlag.

Topsoe:70book

Topsøe, F. (1970). *Topology and Measure*, Volume 133 of *Springer Lecture Notes in Mathematics*. New York: Springer-Verlag.