The duality proof of the SUDAKOV minoration

1 The minoration as a a bound on packing numbers

S:minoration

According to Dudley (2014, p. 131), the following result was first proved by Sudakov (1973, Prop. 7) under a mild extra condition, although credit for similar ideas should be shared with Chevet (1970). It is usually referred to as the SUDAKOV minoration. (I think the word minorant is a synonym for 'lower bound'.)

Sudmin <1> **Sudakov minoration.** *If* $Y := (Y_1, Y_2, \ldots, Y_n)$ *has a centered* MVN *distribution with* $\mathbb{P}|Y_j - Y_k|^2 > \delta^2$ *for all distinct* $j \neq$ *and* k, *then* $\mathbb{P}\max_{i\le n} Y_i \ge C_{sud}\delta\sqrt{\log_2 n}$ *with* C_{sud} *a universal (positive) constant.*

> **Remark.** I found it helpful to keep in mind the extreme cases:
> (i) $Y_j = j\epsilon Z$ for $j = 1, \ldots, n$, where $\epsilon > \delta$ and $Z \sim N(0,1)$;
> (ii) $Y_1, \ldots, Y_n$ are iid $N(0, \sigma^2)$ with $2\sigma^2 > \delta^2$.

The inequality can be regarded as an upper bound for packing numbers or a lower bound for $\mathbb{P}\max_{i\le n} Y_i$. It is also equivalent to an assertion involving covering numbers of convex bodies in euclidean space, the ***Main theorem*** in this note. Talagrand (Ledoux and Talagrand, 1991, pp.82–84,88) claimed credit for the form of the proof.

Sudmin2 <2> **Main theorem.** *Suppose K is a compact, convex, symmetric subset of $\mathbb{R}^k$ with $0 \in \text{interior}(K)$. Then*

\E@ cover.lower <3>
$$\mathcal{F} := \gamma_k \sup_{t\in K}\langle t, g\rangle \ge \sup_{\epsilon>0} c\epsilon\sqrt{\log \text{COVER}(\epsilon, K, \ell^2)}$$

for some universal, positive constant c, where γ_k denotes the $N(0, I_k)$ probability measure.

Remarks.

(i) Here ℓ^2 denotes the usual euclidean metric on $\mathbb{R}^k$, which corresponds to the inner product $\langle x, y\rangle := \sum_{i\le k} x_i y_i$ and norm $|x|_2 := \sqrt{\langle x, x\rangle}$.

(ii) The covering number in <3> could be replaced by a packing number by virtue of the inequalities

\E@ cover.pack <4>
$$\begin{aligned}\text{COVER}_T(\delta, S, d) &\le \text{COVER}_S(\delta, S, d)\\ &\le \text{PACK}(\delta, S, d)\\ &\le \text{COVER}_T(\delta/2, S, d) \le \text{COVER}_S(\delta/2, S, d)\end{aligned}$$

for any totally bounded subset S of any metric space (T, d). In general it matters little whether the centers of the covering balls are restricted to lie in S, as in $\text{COVER}_S(\dots)$, or are allowed to be outside S, as in $\text{COVER}_T(\dots)$. When an argument depends on having centers in S, as in Lemma <6>, it can help to use packing numbers.

(iii) If $\text{diam}(K)$ denotes the ℓ^2 diameter of K then the supremum on the right-hand side of <3> need only be taken over $0 < \epsilon < \text{diam}(K)$ because $B[t, r] \supset K$ if $t \in K$ and $r \ge \text{diam}(K)$.

2 The geometry of the minoration

S:geometry

Let K be a compact, convex, symmetric subset of $\mathbb{R}^k$ with $0 \in \text{interior}(K)$, as in Theorem <2>. Then the closed, convex set

$$L := \{y \in \mathbb{R}^k : \langle y, x\rangle \le 1 \text{ for every } x \text{ in } K\} = \cap_{x\in K}\{y : \langle y, x\rangle \le 1\}$$

is called the ***polar*** of K. It is also compact and symmetric with 0 an interior point. Schneider (1993, §1.6) denoted the polar by K^*; Barvinok (2002, p. 143) denoted it by K^o.

The ***gauge function*** ρ_K is defined on $\mathbb{R}^k$ by

$$\rho_K(x) := \inf\{\lambda > 0 : x/\lambda \in K\}.$$

It is actually a norm on $\mathbb{R}^k$ for which K is the unit ball, that is, $K = \{x : \rho_K(x) \le 1\}$. The ***support function*** h_K is defined on $\mathbb{R}^k$ by

$$h_K(y) := \sup_{x\in K}\langle x, y\rangle.$$

It is very convenient that polar of L equals K and $\rho_L = h_K$ and $\rho_K = h_L$. See Section 4. These facts imply

\E@ rho.rho <5>
$$\langle x, y\rangle \le \rho_K(x)\rho_L(y) \qquad \text{for all } x, y \text{ in } \mathbb{R}^k.$$

If either $\rho_K(x)$ or $\rho_L(y)$ is zero the result is trivial, otherwise $x/\rho_K(x) \in K$ and $y/\rho_L(y) \in L$ so that their inner product is ≤ 1.

3 Proof of the main theorem

S:ProofSudmin2

The following arguments are based on van Handel (2016, pp. 159–162) and Vershynin (2019, pp. 46–50).

We now have three different norms on $T := \mathbb{R}^k$, namely $|\cdot|_2$ and ρ_K and ρ_L. Their unit balls centered at the origin are B and K and L. More generally,

$$\begin{aligned} B[x,r] &:= \{y \in T : |y-x|_2 \le r\}, \\ K[x,r] &:= \{y \in T : \rho_K(y-x) \le r\}, \\ L[x,r] &:= \{y \in T : \rho_L(y-x) \le r\}. \end{aligned}$$

The following three inequalities involving these covering numbers contain the central idea in the proof of the Theorem.

packBL <6> **Lemma.** $\text{COVER}_T(\delta, B, \rho_L) \le 2\exp\left(8\mathfrak{F}^2/\delta^2\right)$, *where* $\mathfrak{F} := \gamma_k \sup_{t\in K}\langle t, g\rangle$, *for each* $\delta > 0$.

packKL <7> **Lemma.** $\text{COVER}_T(\delta, K, \ell^2) \le \text{COVER}_T(\delta^2/4, K, \rho_L)$ *for each* $\delta > 0$.

del.2del <8> **Lemma.** $\text{COVER}(\delta, K, \ell^2) \le \text{COVER}(2\delta, K, \ell^2) \times \text{COVER}_T(\delta/8, B, \rho_L)$ *for each* $\delta > 0$.

Let me first show how these inequalities lead to inequality <3> and then prove the three Lemmas. To simplify notation, define $N(\delta) := \text{COVER}_T(\delta, K, \ell^2)$.

We need to find a positive constant C for which

$$f(\epsilon) := \epsilon\sqrt{\log N(\epsilon)} \le C\mathfrak{F} \qquad \text{for } 0 < \epsilon < D := \text{diam}(K).$$

By Problem [1] we have $\sqrt{2\pi}\mathfrak{F} \ge D := \text{diam}(K)$. Lemma <8> gives

$$\sqrt{\log N(\epsilon)} \le \sqrt{\log N(2\epsilon)} + \sqrt{\log \text{COVER}(\epsilon/8, B, \rho_L)}.$$

And, for $0 < \epsilon \le D$, Lemma <6> then gives

$$\epsilon\sqrt{\log \text{COVER}(\epsilon/8, B, \rho_L)} \le D\sqrt{\log 2} + \epsilon\sqrt{8^3\mathfrak{F}^2/\epsilon^2} \le M,$$

where M is constant multiple of $\mathfrak{F}$. Thus

\E@ xx <9>
$$f(\epsilon) \le f(2\epsilon)/2 + M \qquad \text{for } 0 < \epsilon \le D.$$

Define $R_i := D/2^i$ for $i = 0, 1, 2, \dots$. As already noted, $f(R_0) = 0$. Monotonicity of $\epsilon \mapsto \text{COVER}(\epsilon, K, \ell^2)$ gives

$$f(\epsilon) \le R_i\sqrt{\log \text{COVER}(R_{i+1}, K, \ell^2)} = 2f(R_{i+1}) \qquad \text{for } R_{i+1} \le \epsilon \le R_i.$$

Thus it remains only to derive an upper bound for $f(R_i)$ from the inequality

$$f(R_{i+1}) \le f(R_i)/2 + M.$$

Iterate.

$$
\begin{aligned}
&f(R_1) \leq 0/2 + M, \\
&f(R_2) \leq f(R_1)/2 + M \leq M/2 + M, \\
&f(R_3) \leq f(R_2)/2 + M \leq M/4 + M/2 + M,
\end{aligned}
$$

and so on. We must have $f(R_i) \leq 2M$ for all i. Done.

The proofs for the Lemmas are quite short.

Proof (of Lemma <6>). The assertion is

$$\text{COVER}_T(\delta, B, \rho_L) \leq 2\exp\left(8\mathcal{F}^2/\delta^2\right).$$

It suffices (cf. <4>) to show that $N := \text{PACK}(\delta, B, \rho_L) \leq \mathcal{N}(\delta)$.

Let $t_1, \dots, t_N$ be points of B for which $\rho_L(t_i - t_j) > \delta$. Define $x_i := Rt_i$ where $R := 4\mathcal{F}/\delta$. The set $\{x_1, \dots, x_N\}$ is a $4\mathcal{F}$-separated subset of $B[0, R]$. The ρ_L-balls $L[x_i, 2\mathcal{F}]$ for $i = 1, \dots, N$ are disjoint. Thus

$$1 \geq \sum\nolimits_i \gamma_k L[x_i, 2\mathcal{F}] \geq N \min\nolimits_{i \leq N} \gamma_k L[x_i, 2\mathcal{F}].$$

We need to show $\gamma_k L[x, 2\mathcal{F}] \geq \exp(-R^2/2)/2$ when $x \in B[0, R]$. Start with the case $A := L[0, 2\mathcal{F}]$. Define $W := \sup_{t \in K}\langle t, g\rangle = h_K(g) = \rho_L(g)$. Then we have

$$\gamma_k A^c = \gamma_k\{\rho_L(g) > 2\mathcal{F}\} = \gamma_k\{W > 2\mathcal{F}\} \leq \gamma_k W/(2\mathcal{F}) = 1/2.$$

Hence $\gamma_k A \geq 1/2$. Write P for the conditional distribution $\gamma_k(\cdot \mid A)$, which has density $\{y \in A\}\phi(y)/\gamma_k A$ with respect to lebesgue measure, where $\phi(y) := (2\pi)^{-k/2}\exp(-|y|_2^2/2)$. For each x in $\mathbb{R}^k$ we have

$$
\begin{aligned}
\gamma_k L[x, 2\mathcal{F}] &= \int \{y - x \in A\}\phi(y)\,dy = \int \{w \in A\}\phi(w + x)\,dw \\
&= \int \{w \in A\}\exp\left[-|x|_2^2/2 - \langle x, w\rangle\right]\phi(w)\,dw \\
&= (\gamma_k A)\exp\left(-|x|_2^2/2\right) P^w \exp\left(-\langle x, w\rangle\right) \\
&\geq \tfrac{1}{2}\exp\left(-|x|_2^2/2\right)\exp\left(-\langle x, Pw\rangle\right) \qquad \text{by the Jensen inequality,} \\
&\geq \tfrac{1}{2}\exp\left(-|x|_2^2/2\right) \qquad \text{because } Pw = 0 \text{ by symmetry.}
\end{aligned}
$$

$\square$ In particular, if $x \in B[0, R]$ then $\gamma_k L[x, 2\mathcal{F}] \geq \exp(-R^2/2)/2$.

Proof (of Lemma <7>). The assertion is

$$\text{COVER}_T(\delta, K, \ell^2) \leq \text{COVER}_T(\delta^2/4, K, \rho_L).$$

Define $\epsilon := \delta^2/2$. This time let $F := \{x_1, \dots, x_N\}$, for $N = \text{PACK}(\epsilon, K, \rho_L)$, be a maximal ϵ-separated (in ρ_L distance) subset of K. Then $K \subset \cup_{i \leq N} L[x_i, \epsilon]$. If $y \in K \cap L[x_i, \epsilon]$ then $\rho_K(y - x_i) \leq \rho_K(y) + \rho_K(x_i) \leq 2$ and $\rho_L(y - x_i) \leq \epsilon$ so that

$$|y - x_i|_2^2 \leq \rho_K(y - x_i)\rho_L(y - x_i) \leq 2\epsilon \qquad \text{by inequality <5>.}$$

It follows that $y \in B[x_i, \sqrt{2\epsilon}\,] = B[x_i, \delta]$. Hence

$$K = \cup_{i \le N} \left(K \cap L[x_i, \epsilon]\right) \subset \cup_{i \le N} B[x_i, \delta],$$

□ implying $\text{COVER}_K(\delta, K, \ell^2) \le N \le \text{COVER}(\epsilon/2, K, \ell^2)$.

Proof (of Lemma <8>). The assertion is

$$\text{COVER}_T(\delta, K, \ell^2) \le \text{COVER}_T(2\delta, K, \ell^2) \times \text{COVER}_T(\delta/8, B, \rho_L).$$

The argument involves two coverings: first choose $x_1, \dots, x_N$ with $N := \text{COVER}_T(2\delta, K, \ell^2)$ for which $K \subset \cup_{i \le N} B[x_i, 2\delta]$; then choose $z_1, \dots, z_M$ with $M := \text{COVER}_T(r, B, \rho_L)$ for which $\min_j \rho_L(w - z_j) \le r := \delta/8$ for each w in B. In particular, if $y \in B[x_i, 2\delta]$ there there exists a j for which

$$\rho_L\left(\frac{y - x_i}{2\delta} - z_j\right) \le r \qquad \text{and thus} \qquad \rho_L\left(y - x_i - 2\delta z_j\right) \le 2\delta r = \delta^2/4,$$

which implies

$$K \subset \cup_{i \le N} B[x_i, 2\delta] \subset \cup_{i \le N} \cup_{j \le M} L[x_i + 2\delta z_j, \delta^2/4].$$

It follows, via Lemma <7>, that

□ $$\text{COVER}_T(\delta, K, \ell^2) \le \text{COVER}_T(\delta^2/4, K, \rho_L) \le NM$$

4 Facts about duality

S:polar

Sources: E = Eggleston (1958, §1.9, §3,5) and S = Schneider (1993, §1.6, §1.7).

S-p8 defined

$$\mathcal{K}^n := \text{ all nonempty, compact, convex subsets of } \mathbb{R}^n ,$$
$$\mathcal{K}^n_o := \{K \in \mathcal{K}^n : \text{interior}(K) \ne \emptyset\}.$$

He called members of $\mathcal{K}^n$ ***compact bodies***.

The ***support function*** can be defined for a (nonempty) bounded subset A of $\mathbb{R}^n$ as

$$h(u, A) := h_A(u) := \sup\{\langle x, u\rangle : x \in A\} \qquad \text{for } u \in \mathbb{R}^n.$$

(S used nonempty closed convex sets, initially allowing h to take the value $+\infty$.) If A is closed (and hence compact) the supremum is achieved for some x_0 in A. In that case, the supporting hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = h_A(u)\}$ contains the point x_0 and the closed halfspace $\{x \in \mathbb{R}^n : \langle x, u\rangle \le h_A(u)\}$ contains A.

The ***polar*** of a bounded set A, denoted by A^o or A^*, is defined as

$$\begin{aligned} A^* &:= \{y \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } x \text{ in } A\} \\ &= \cap_{x \in A}\{y \in \mathbb{R}^n : \langle x, y\rangle \le 1\} \qquad \text{so } A^* \text{ is closed and convex} \\ &= \{y \in \mathbb{R}^n : h_A(y) \le 1\}. \end{aligned}$$

If $A \subseteq B[\mathbf{0}, R]$ then $|h_A(u)| \le R|u|_2$ for every u in $\mathbb{R}^n$, which implies $B[0, 1/R] \subset A^*$. In particular, $\mathbf{0}$ is an interior point of A^* and $A^* \in \mathcal{K}^n_o$. Similarly, if the origin is an interior point of A then $B[\mathbf{0}, \delta] \subset A$ for some $\delta > 0$. For each y in A^* we then have $\langle x, y\rangle \le 1$ for each x with $|x|_2 \le \delta$, which implies $|y|_2 \le 1/\delta$ and $A^* \subset B[0, 1/\delta]$.

Remark. E-p25 allowed A to be any subset of $\mathbb{R}^n$. If A is unbounded then interior(A^*) can be empty. S-p3 defined the polar only for members of $\mathcal{K}^n_o$.

The polar of the polar, $A^{**} := \{z \in \mathbb{R}^n : \langle z, y\rangle \le 1 \text{ for all } y \text{ in } A^* \}$ also belongs to $\mathcal{K}^n$. Clearly $A^{**} \supset A \cup \{\mathbf{0}\}$. E-p25 claimed that A^{**} is actually the convex hull of $S := \overline{A} \cup \{\mathbf{0}\}$.

polar2 <10> **Lemma.** *If $K \in \mathcal{K}^n_o$ with $\mathbf{0}$ as an interior point then $K^{**} = K$.*

Proof. As $\mathbf{0} \in$ interior(K) the polar K^* is bounded; it also belongs to $\mathcal{K}^n_o$. The inequality

$$\langle x, y\rangle \le 1 \qquad \text{for all } x \in K \text{ and all } y \in K^*$$

implies that $K \subset K^{**}$.

To complete the proof it suffices to show that if $w \notin K$ then $w \notin K^{**}$. Such a w can be separated from K by a hyperplane: for some $u \in \mathbb{R}^n$ and some $\alpha \in \mathbb{R}$,

$$\langle w, u\rangle > \alpha > \sup_{x\in K}\langle x, u\rangle.$$

The constant α must be > 0 because $\mathbf{0} \in K$. The fact that $1 \ge \langle x, u/\alpha\rangle$ for each x in K shows that $u/\alpha \in K^*$. The fact that $\langle w, u/\alpha\rangle > 1$ then implies $w \notin (K^*)^*$. □

The ***gauge function*** of a set K in $\mathcal{K}^n_o$ is defined by S-p43 as

$$g(x, K) := \inf\{\lambda \ge 0 : x \in \lambda K\}$$

The infimum is achieved by some $\lambda > 0$ if $x \ne \mathbf{0}$, that is, $g(x, K)$ is the smallest $\lambda > 0$ for which $x/\lambda \in K$. For K^* and $y \ne \mathbf{0}$ we have

$$\begin{aligned} g(y, K^*) &= \min\{\lambda > 0 : \langle y/\lambda, x\rangle \le 1 \text{ for all } x \in K \ \} \\ &= \min\{\lambda > 0 : \langle y, x\rangle \le \lambda \text{ for all } x \in K \ \} = h(y, K). \end{aligned}$$

Similarly,

$$g(x, K) = g(x, K^{**}) = h(x, K^*).$$

5 Equivalence between the two forms of the minoration

S:equivalence

Theorem <2> involved a set of functions $X = \{X_t : t \in K\}$ indexed by a subset K of $\mathbb{R}^k$. More precisely, the function X_t was defined by $X_t(g) := \langle t, g\rangle = \sum_{i\le k} t_i g_i$ with g a generic point of $\mathbb{R}^k$. Under the probability measure $\gamma_k := N(0, I_k)$ on $\mathbb{R}^k$, the funtion X_t became a random variable and X became a centered gaussian process with

$$\text{cov}\,(X_s, X_t) = \langle s, t\rangle \quad \text{AND} \quad \text{var}(X_t) = |t|_2^2.$$

Inequality <3> could be rewritten using packing numbers:

\E@ Sudmin3 <11>

$$C\mathbb{P}\sup_{t\in K} X_t \ge \epsilon\sqrt{\log \text{PACK}(\epsilon, K, \ell^2)} \qquad \text{for each } \epsilon > 0.$$

If, for some given $\epsilon > 0$, we have a finite subset F of K with $|s - t|_2 > \epsilon$ then

$$\|X_s - X_t\|_2 = |s - t|_2 > \epsilon \qquad \text{for distinct } s, t \text{ in } F.$$

Thus <11> is a special case of Theorem <1>.

It takes a little more work to embed a random vector $Y = (Y_1, \dots, Y_n)$ with a $N(0, V)$ distribution into a process indexed by K.

First note that $\mathfrak{F}_Y := \mathbb{P}\max_i Y_i = \mathbb{P}\max_i(Y_i - Y_1)$ and

$$\|Y_i - Y_j\|_2 = \|(Y_i - Y_1) - (Y_j - Y_1)\|_2\,.$$

Thus we lose no generality by assuming Y_1 is identically 0.

Remark. This little trick eliminates an annoying difficulty when a maximum over a finite set needs to be bounded by a sum of maxima over two subsets.

If we define $Z = (Z_1, \dots, Z_n)$ by $Z_i := -Y_i$ then $Z \sim N(0, V)$, so that

$$\max\{Y_1, \dots, Y_n, Z_1, \dots, Z_n\} \le \max_i Y_i + \max_i Z_i.$$

It helps that both $\max_i Y_i$ and $\max_i Z_i$ are nonnegative. It follows that

$$\mathfrak{F}_{Y,Z} := \mathbb{P}\max\{Y_1, \dots, Y_n, Z_1, \dots, Z_n\} \le \mathbb{P}\max_i Y_i + \mathbb{P}\max_i Z_i = 2\mathfrak{F}_Y.$$

The quantity $\mathfrak{F}_Y$ depends only on the distribution of Y. It helps to work with a specific representation of that distribution. Suppose $\text{RANK}(V) = k$. (Necessarily $k < n$ because I made Y_1 identically zero.) Then there exists a $k \times n$ matrix $A = [a_1, \dots, a_n]$ with $V = A^\intercal A$ and $\text{RANK}(A) = k$.

Remark. Such an A could be constructed from a spectral representation, $V = U\text{diag}(\lambda_1, \dots, \lambda_n)U^\intercal$, with U an orthogonal matrix and only k of the λ_i's nonzero.

Under γ_k, the vector $(\langle a_1, g\rangle, \dots, \langle a_n, g\rangle)$ has a $N(0, V)$ distribution. We could take Y_i to equal $\langle a_i, g\rangle$ and Z_i equal to $\langle -a_i, g\rangle$.

Define K to be the convex hull of the set of

$$E := \{a_1, \dots, a_n, -a_1, \dots, -a_n\}.$$

Then K is a compact, convex, symmetric subset of $\mathbb{R}^k$ with $0 \in K$. The extreme points of K all belong to E. It is not too hard (Problem [2]) to deduce from $\text{RANK}(A) = k$ that 0 is an interior point.

For each g the linear functional $t \mapsto \langle t, g\rangle$ achieves its maximum over K at extreme point in E. Thus

$$\gamma_k \sup_{t\in K}\langle t, g\rangle = \gamma_k \sup_{t\in E}\langle t, g\rangle = \mathfrak{F}_{Y,Z} \le 2\mathfrak{F}_Y.$$

6 Problems

S:Problems

P:bdd.paths [1] Suppose $\{X_t : t \in T\}$ is a DOOB-separable, centered gaussian process.

(i) For each pair s, t in T show that

$$2\mathbb{P}\max(X_s, X_t) = \mathbb{P}\left(|X_s - X_t| + X_s + X_t\right) = 2\left\|X_s - X_t\right\|_2 / \sqrt{2\pi}.$$

Hint: If $Z \sim N(0, 1)$ then $\mathbb{P}|Z| = 2/\sqrt{2\pi}$.

(ii) Deduce that $\mathbb{P}\sup_{t\in T} X_t \ge \sup_{s,t\in T}\left\|X_s - X_t\right\|_2 / \sqrt{2\pi}$.

P:interior [2] Using the notation from Section 5, prove that 0 is an interior point of K by arguing as follows.

(i) The fact that A has rank equal to k implies that the set $\{a_j : j \in J\}$ is a basis for $\mathbb{R}^k$ for some subset J of $[[n]]$.

(ii) Let $\{e_\alpha : \alpha \in [[k]]\}$ denote the usual orthonormal basis for $\mathbb{R}^k$. For each α there are real numbers $\theta[\alpha, j]$ for which $e_\alpha = \sum_{j\in J}\theta[\alpha, j]a_j$. Define M to be the positive square root of $\max_\alpha \sum_{j\in J}\theta[\alpha, j]^2$. Suppose $x = \sum_\alpha r_\alpha e_\alpha \in B[0, \delta]$. Then $x = \sum_{j\in J}\lambda_j a_j$ with $|\lambda_j| \le |\sum_\alpha r_\alpha \theta[\alpha, j]| \le \delta M$. Thus $\sum_{j\in J}|\lambda_j| \le 1$ if δ is small enough.

(iii) Thus $x = \sum_{j\in J}|\lambda_j|\text{SGN}(\lambda_j)a_j \in K$.

References

Barvinok2002book Barvinok, A. (2002). *A Course in Convexity*, Volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society.

Chevet1970 Chevet, S. (1970). Mesures de Radon sur $\mathbf{R}^n$ et mesures cylindriques. *Annales scientifiques de l'Université de Clermont. Mathématiques 43*(6), 91–158. Available at http://eudml.org/doc/80390.

Dudley2014UCLT Dudley, R. M. (2014). *Uniform Central Limit Theorems* (2nd ed.), Volume 142 of *Cambridge studies in advanced mathematics*. Cambridge University Press. (First edition, 1999).

Eggleston1958Convexity Eggleston, H. G. (1958). *Convexity*. Number 47 in Cambridge Tracts in Mathematics. Cambridge University Press.

LedouxTalagrand91book Ledoux, M. and M. Talagrand (1991). *Probability in Banach Spaces: Isoperimetry and Processes*. New York: Springer.

Schneider1993 Schneider, R. (1993). *Convex Bodies: The Brunn-Minkowski Theory*. Number 44 in Encyclopedia of Mathematics and its Applications. Cambridge University Press.

Sudakov1973minoration Sudakov, V. (1973). A remark on the criterion of continuity of Gaussian sample function. In *Proceedings of the Second Japan-USSR Symposium on Probability Theory, Kyoto, August 2–9, 1972*, Volume 330 of *Springer Lecture Notes in Mathematics*, pp. 444–454.

vanHandel2016PiHD van Handel, R. (2016, December). Probability in high dimension. APC 550 Lecture Notes, Princeton University.

Vershynin2019GFA Vershynin, R. (2019, May). Lectures in geometric functional analysis. Available online at https://www.math.uci.edu/∼rvershyn/.