R-THEORY FOR MARKOV CHAINS ON A TOPOLOGICAL STATE SPACE I

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1. *Introduction*

If $\{X_n\}$ is a discrete-time ϕ-irreducible Markov chain on a measure space $(\mathscr{X}, \mathscr{F})$ [4; p. 4], with n-step transition probabilities $P^n(x, A)$, it has been shown in [5] that there exists a subset $\mathscr{C}_R$ of $\mathscr{F}$ with the property that, for every $A \in \mathscr{C}_R$ and ϕ-almost all $x \in \mathscr{X}$, the power series $\sum P^n(x, A)z^n$ have the same radius of convergence R. Moreover, there is a countable partition of $\mathscr{X}$ all of whose elements belong to $\mathscr{C}_R$.

If all the power series diverge for $z = R$, and $\{X_n\}$ is aperiodic, then there is a second subset $\mathscr{C}_L$ of $\mathscr{F}$ such that for any $A \in \mathscr{C}_L$, $\lim_{n\to\infty} P^n(x, A)R^n = \pi(x, A) < \infty$ exists for almost all $x \in \mathscr{X}$. The state space $\mathscr{X}$ can again be countably partitioned into elements of $\mathscr{C}_L$.

In this paper we assume that a topology exists on $\mathscr{X}$, and investigate continuity conditions on the transition probabilities of $\{X_n\}$ which will ensure that compact elements of $\mathscr{F}$ lie in either $\mathscr{C}_R$ or $\mathscr{C}_L$. A condition sufficient for both these desirable attributes is given in §3, and in §4 we consider weakening this condition. Examples are given to show that under the weaker condition compact sets may or may not belong to $\mathscr{C}_R$ or $\mathscr{C}_L$, and some auxiliary conditions on $\mathscr{X}$ are found which make the weaker continuity conditions sufficient for compact sets to belong to $\mathscr{C}_R$ and $\mathscr{C}_L$.

2. *Preliminaries*

We assume as usual that $P^n(x, A) = \Pr\{X_n \in A \mid X_0 = x\}$, $A \in \mathscr{F}$, $x \in \mathscr{X}$, is for each x a probability measure on $\mathscr{F}$ and for each $A \in \mathscr{F}$ a measurable function on $\mathscr{X}$. We write

$$G_z(x, A) = \sum_{n=1}^{\infty} P^n(x, A) z^n$$

for the generating functions of these probabilities. We also assume that $\{X_n\}$ satisfies

CONDITION I: *There exists a non-trivial finite measure M on $\mathscr{F}$ such that*

(i) $\{X_n\}$ *is M-irreducible; that is, whenever $M(A) > 0$, then $G_{\frac{1}{2}}(x, A) > 0$ for all* $x \in \mathscr{X}$;

(ii) *whenever $M(A) = 0$, then $M\{y : G_{\frac{1}{2}}(y, A) > 0\} = 0$.*

It is shown in [5] that this assumption is equivalent to ϕ-irreducibility for some ϕ. We use M to denote a fixed measure satisfying Condition I, and unless otherwise qualified, such phrases as " almost everywhere " will refer to M-measure. We write $\mathscr{F}^+ = \{A \in \mathscr{F}, M(A) > 0\}$.

The following results are proved in [5].

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THEOREM A. *There exists a real number $R^{-1} \leqslant 1$ (the convergence norm of $\{X_n\}$), a null set $N \in \mathscr{F}$, and a subset $\mathscr{C}_R$ of $\mathscr{F}^+$ such that*

(i) *for every $x \notin N$ and $A \in \mathscr{C}_R$, the radius of convergence of $G_z(x, A)$ is R.*

(ii) *There is a countable partition $(K(j), j = 1, 2, \ldots)$ of $\mathscr{X}$ with $K(j) \in \mathscr{C}_R$ for each j.*

(iii) *Either $G_R(x, A) = \infty$ for every $x \notin N$ and $A \in \mathscr{C}_R$, and $\{X_n\}$ is called R-recurrent; or $G_R(x, A) < \infty$ for every such x and A, and $\{X_n\}$ is called R-transient.*

If $\{X_n\}$ is R-recurrent, then its behaviour mimics closely that of ordinary recurrent chains, and the theory of recurrent chains is often subsumed in that of R-recurrent chains. One aspect so subsumed is the existence of subinvariant measures and functions. For general $r > 0$, an *r-subinvariant measure for $\{X_n\}$* is a σ-finite, non-trivial measure μ on $\mathscr{F}$ satisfying, for every $A \in \mathscr{F}$,

$$\mu(A) \geqslant r \int_{\mathscr{X}} \mu(dy)\, P(y, A); \tag{2.1}$$

an *r-subinvariant function for $\{X_n\}$* is a non-negative, measurable function g on $\mathscr{X}$, with $\{y : g(y) > 0\} \in \mathscr{F}^+$, satisfying, for almost all $x \in \mathscr{X}$,

$$g(x) \geqslant r \int_{\mathscr{X}} P(x, dy)\, g(y). \tag{2.2}$$

THEOREM B. *If $\{X_n\}$ is R-recurrent, then there exist a unique (up to constant multiples) R-subinvariant measure Q, which is R-invariant (satisfies (2.1) with equality for every A) and is equivalent to M, and a unique (up to constant multiples and definition on null sets) R-subinvariant function f, which is R-invariant (satisfies (2.2) with equality for almost all x).*

We shall use Q and f exclusively to denote the unique R-invariant measure and function for $\{X_n\}$ when $\{X_n\}$ is R-recurrent. If $N_f^{(1)}$ is the set where f fails to satisfy (2.2), and $N_f^{(2)} = \{y : G_{\frac{1}{2}}(y, N_f^{(1)}) > 0\}$, we write $N_f = N_f^{(1)} \cup N_f^{(2)}$: from Condition I (ii) and Theorem B, $M(N_f) = 0$.

THEOREM C. *If $\{X_n\}$ is R-recurrent and aperiodic* [4; p. 15], *then there exists a subset $\mathscr{C}_L \subseteq \mathscr{F}^+$ such that*

(i) *for $A \in \mathscr{C}_L$, there exists N_A with $M(N_A) = 0$ such that*

$$\pi(x, A) = \lim_{n \to \infty} P^n(x, A) R^n$$

exists and is finite for all $x \notin N_A$.

(ii) *There is a countable partition of $\mathscr{X}$ each of whose elements is in $\mathscr{C}_L$.*

(iii) *The set N_A can be chosen so that for $x \notin N_A$,*

$$\pi(x, A) = f(x)\, Q(A) \Big/ \int_{\mathscr{X}} f(y)\, Q(dy). \tag{2.3}$$

(iv) *Either*

$$\int_{\mathscr{X}} f(y)\, Q(dy) < \infty,$$

in which case $\pi(x, A) > 0$ *for all* $x \notin N_A$ *and* $A \in \mathscr{C}_L$, *and* $\{X_n\}$ *is called R-positive; or*

$$\int_{\mathscr{X}} f(y)\, Q(dy) = \infty,$$

and $\pi(x, A) \equiv 0$, $x \notin N_A$, $A \in \mathscr{C}_L$, *when* $\{X_n\}$ *is called R-null.*

Neither $\mathscr{C}_R$ nor $\mathscr{C}_L$ is uniquely defined by the above theorems. Let us call a subset $\mathscr{C}$ of $\mathscr{F}^+$ an *R-system* if it satisfies Theorem A and an *L-system* if it satisfies Theorem C, and a set $A \in \mathscr{F}^+$ an *R-set* if it is contained in an R-system and an *L-set* if it is contained in an L-system. If A is an L-set, we use N_A to denote the null set in Theorem C, and always assume that N_A contains both $\{y : G_{\frac{1}{2}}(y, N_A) > 0\}$ and N_f. The following propositions give explicit constructions for R-systems and L-systems which will be much used in the sequel.

PROPOSITION 2.1. *Suppose* μ *is an R-subinvariant measure for* $\{X_n\}$. *Then any* $A \in \mathscr{F}^+$ *such that* $\mu(A) < \infty$ *is an R-set, and* $\{A \in \mathscr{F}^+ : \mu(A) < \infty\}$ *is an R-system.*

Proof. In the nomenclature of [5], an R-set is an R-recurrent set if $\{X_n\}$ is R-recurrent and an R-transient set if $\{X_n\}$ is R-transient. The first statement is thus merely a rephrasing of Proposition 10.3 of [6] and its corollary, and the second follows since μ is σ-finite.

PROPOSITION 2.2. *Suppose* $\{X_n\}$ *is R-recurrent, with R-invariant measure* Q *and function* f. *Then any set* $A \in \mathscr{F}$ *such that*

$$0 < Q(A) < \infty \tag{2.4a}$$

$$\inf\{f(x), x \in A \setminus N_A\} > 0 \tag{2.4b}$$

is an L-set, and the set of elements of $\mathscr{F}$ *satisfying* (2.4) *is an L-system.*

Proof. Since Q is equivalent to M, and σ-finite, (2.4a) implies that $A \in \mathscr{F}^+$; and there is a partition of $\mathscr{X}$ on each element of which (2.4) holds.

In the R-positive case, the sufficiency of (2.4) is implicit in the proof of Theorem 6 of [5]. In the R-null case, the proof given of that theorem needs the assumption that A satisfies (2.4b) and, rather than (2.4a), the finiteness condition

$$\int_A Q(dy) f(y) < \infty.$$

This condition (stronger than (2.4a) when (2.4b) holds) can be removed by imitating directly the proof of Theorem 7.3 of [4], to prove that if A satisfies (2.4), and $\{X_n\}$ is R-null, then $R^n P^n(x, A) \to 0$ for almost all x.

Save mentioning that (2.4a) is needed to assume the analogue of (7.6) in [4], whilst (2.4b) enables one to prove the R-theoretic analogue of the Corollary to Theorem 5.1 in [4] (in a manner similar to its use in the proof of Theorem 6 in [5]), we omit the details.

3. *The sufficiency of strong continuity*

From now on we shall assume that the set $\mathscr{X}$ is equipped with a topology $\mathscr{T}$, and shall seek conditions which ensure that $\mathscr{T}$-compact sets are R-sets or L-sets. We shall write $\mathscr{K}$ for the set of compact elements of $\mathscr{F}^+$.

The set of transition probabilities $\{P(x, \cdot), x \in \mathscr{X}\}$ will be called *strongly continuous* if, for any fixed A, $P(x, A)$ is a continuous function of x: that is, for every net $\{x_\alpha\}$ of points in $\mathscr{X}$ converging to a point $x \in \mathscr{X}$, we have

$$P(x_\alpha, A) \to P(x, A), \text{ all } A \in \mathscr{F}. \tag{3.1}$$

Let $\mathbf{D}(\mathscr{X})$ be the set of bounded measurable functions on $\mathscr{X}$, and let $\mathbf{C}(\mathscr{X})$ be the set of continuous functions in $\mathbf{D}(\mathscr{X})$. It is well known that (3.1) is equivalent to

$$\int_{\mathscr{X}} P(x_\alpha, dy)\, g(y) \to \int_{\mathscr{X}} P(x, dy)\, g(y), \text{ all } g \in \mathbf{D}(\mathscr{X}); \tag{3.2}$$

that is, the map

$$g \to \int_{\mathscr{X}} P(\cdot, dy)\, g(y)$$

takes $\mathbf{D}(\mathscr{X})$ into $\mathbf{C}(\mathscr{X})$. A chain satisfying (3.2) is often called *strongly Feller* (*cf.* [**2**; p. 58]).

In this section we prove

THEOREM 1. *Suppose $\{P(x, \cdot)\}$ is strongly continuous. Then every $K \in \mathscr{K}$ is both an R-set and an L-set.*

The proof of Theorem 1 consists in verifying the conditions of Propositions 2.1 and 2.2. We give these verifications as a sequence of propositions.

PROPOSITION 3.1. *If $\{P(x, \cdot)\}$ is strongly continuous, then so is $\{G_\theta(x, \cdot)\}$, for any $\theta < 1$.*

Proof. Suppose inductively that $\{P^n(x, \cdot)\}$ is strongly continuous. For any $g \in \mathbf{D}(\mathscr{X})$,

$$\int P^{n+1}(\cdot, dy)\, g(y) = \int P(\cdot, dw)\, [\int P^n(w, dy)\, g(y)] \in \mathbf{C}(\mathscr{X}),$$

since $\int P^n(\cdot, dy)\, g(y) \in \mathbf{C}(\mathscr{X}) \subseteq \mathbf{D}(\mathscr{X})$ and $\{P(x, \cdot)\}$ is strongly continuous. Therefore for all n the set $\{P^n(x, \cdot)\}$ is strongly continuous.

Suppose $\theta < 1$, and let x be the $\mathscr{T}$-limit of $\{x_\alpha\}$. Then for all $A \in \mathscr{F}$

$$\begin{aligned} \lim_\alpha G_\theta(x_\alpha, A) &= \sum_{n=1}^\infty \theta^n \lim_\alpha P^n(x_\alpha, A) \\ &= \sum_{n=1}^\infty \theta^n P^n(x, A) \\ &= G_\theta(x, A), \end{aligned}$$

the first equality by the uniform convergence of the series for $G_\theta(\cdot, A)$, the second from the result above. Thus $\{G_\theta(x, \cdot)\}$ is strongly continuous.

PROPOSITION 3.2. *If $K \in \mathscr{K}$, then $\mu(K) < \infty$ for any R-subinvariant measure μ, when $\{P(x, \cdot)\}$ is strongly continuous. Hence K is an R-set.*

Proof. Iterating (2.1) gives, for each n and any $A \in \mathscr{F}$,

$$\mu(A) \geqslant R^n \int_{\mathscr{X}} \mu(dy)\, P^n(y, A).$$

Multiplying this by β^n, with $\beta < R^{-1}$, and summing, gives

$$\begin{aligned} \frac{\beta}{1-\beta}\, \mu(A) &\geqslant \int_{\mathscr{X}} \mu(dy)\, G_{R\beta}(y, A) \\ &\geqslant \int_K \mu(dy)\, G_{R\beta}(y, A) \\ &\geqslant \mu(K) \inf_{y \in K} G_{R\beta}(y, A). \end{aligned} \tag{3.3}$$

In (3.3), choose A to be any set in $\mathscr{F}^+$ with $\mu(A) < \infty$. From Proposition 3.1, if $\beta < R^{-1}$ the function $G_{R\beta}(y, A)$ is continuous, and hence attains its infimum on compact sets; since $A \in \mathscr{F}^+$ and Condition I imply that $G_{R\beta}(y, A) > 0$ for all y, this gives

$$\inf_{y \in K} G_{R\beta}(y, A) = \delta_K > 0 \tag{3.4}$$

when $K \in \mathscr{K}$. Putting (3.4) in (3.3) gives

$$\mu(K) \leqslant \frac{\beta}{1-\beta} \cdot \frac{\mu(A)}{\delta_K} \tag{3.5}$$

and the proposition holds on applying Proposition 2.1.

The existence of a constant β_K which bounds $\mu(K)/\mu(A)$ for fixed A, as indicated in (3.5), is of interest in the potential theory of R-transient chains, and if $\mathscr{X}$ is σ-compact this proposition enables us to assume that the partition $\mathscr{D}$ occurring throughout [6] can be replaced by any partition consisting of compact sets in $\mathscr{F}^+$ when $\{P(x, \cdot)\}$ is strongly continuous.

Suppose now that $\{X_n\}$ is R-recurrent and aperiodic, with R-invariant measure Q and function f, and that $\{P(x, \cdot)\}$ is strongly continuous. Since Q is equivalent to M, any $K \in \mathscr{K}$ has both $Q(K) > 0$ (since $\mathscr{K} \subseteq \mathscr{F}^+$) and $Q(K) < \infty$ (as a corollary to the previous proposition). To prove that such a K is an L-set, we thus need only verify that K satisfies (2.4b).

If f is R-invariant, and N_f is defined as in §2, we define $S_f = \{y \in \mathscr{X} : P(y, N_f) > 0\}$. Since $S_f \subseteq N_f$, $M(S_f) = 0$; also, S_f is the continuous inverse image of the open set of real numbers $(0, \infty)$, and hence S_f is open. Thus, if $K \in \mathscr{K}$, $K \setminus S_f$ is also a compact set, and $M(K \setminus S_f) = M(K) > 0$; hence $K \setminus S_f$ is again in $\mathscr{K}$.

Define $f^* : \mathscr{X} \setminus S_f \to (0, \infty]$ by

$$\begin{aligned} f^*(x) &= R \int_{\mathscr{X}} P(x, dy) f(y) \\ &= R \int_{\mathscr{X} \setminus N_f} P(x, dy) f(y) \end{aligned} \tag{3.6}$$

since $P(x, N_f) = 0$, $x \notin S_f$. Notice that R-invariance of f ensures that $f^*(x) > 0$ for all $x \in \mathscr{X} \setminus S_f$, and that $f^* = f$ on $\mathscr{X} \setminus N_f$. We prove that f^* is lower semi-continuous on $\mathscr{X} \setminus S_f$(*cf.* [**1**; A6]). Set $A(j) = \{x \in \mathscr{X} \setminus N_f : f(x) \leqslant j\}$: by definition, $A(j) \uparrow \mathscr{X} \setminus N_f$ as $j \to \infty$. Let $\{x_\alpha\}$ be a net on $\mathscr{X} \setminus S_f$ converging to x (which must itself be in $\mathscr{X} \setminus S_f$, since $\mathscr{X} \setminus S_f$ is closed.) We have

$$\begin{aligned} \liminf f^*(x_\alpha) &= \liminf R \int_{\mathscr{X} \setminus N_f} P(x_\alpha, dy) f(y) \\ &\geqslant \liminf R \int_{A(j)} P(x_\alpha, dy) f(y) \\ &= R \int_{A(j)} P(x, dy) f(y) \\ &\to R \int_{\mathscr{X} \setminus N_f} P(x, dy) f(y) \text{ as } j \to \infty \\ &= f^*(x), \end{aligned} \tag{3.7}$$

the equality in the third line being due to the strong continuity of $\{P(x, \cdot)\}$ and the boundedness of f on $A(j)$.

Now (3.7) implies that f^* is lower semi-continuous on $\mathscr{X} \setminus S_f$. If $K \in \mathscr{K}$, then $K \setminus S_f \in \mathscr{K}$, and so (since lower semi-continuous functions attain their infimum on compact sets [**1**; A6], and $f^*(x) > 0$ on $\mathscr{X} \setminus S_f$),

$$\begin{aligned} 0 &< \inf_{y \in K \setminus S_f} f^*(y) \\ &\leqslant \inf_{y \in K \setminus N_f} f^*(y) \\ &= \inf_{y \in K \setminus N_f} f(y). \end{aligned} \tag{3.8}$$

We have thus proved

PROPOSITION 3.3. *If $\{P(x, \cdot)\}$ is strongly continuous and $\{X_n\}$ is aperiodic and R-recurrent, then any $K \in \mathscr{K}$ satisfies $\inf_{x \in K \setminus N_f} f(x) > 0$. Thus K satisfies* (2.4b), *since* $N_K \supseteq N_f$.

This completes the proof of Theorem 1. It is important to note that there are non-trivial cases when $\{P(x, \cdot)\}$ is actually strongly continuous.

Example 1. Let $\{X_n\}$ be a random walk on the real line $\mathbb{R}$, with a density $h(x)$; that is,

$$P(x, A) = \int_A h(y-x)\, dy.$$

Let $g \in L^\infty = \mathbf{D}(\mathbb{R})$; then

$$\begin{aligned} \int_{\mathbb{R}} P(x, dy)\, g(y) &= \int_{\mathbb{R}} h(y-x)\, g(y)\, dy \\ &= h * g(x), \end{aligned} \tag{3.9}$$

where $*$ denotes convolution. Since $\int h(x)\,dx = 1$, (3.9) is the convolution of an L^1 with an L^∞ function, which is actually *uniformly* continuous on $\mathbb{R}$ [3; p. 398]. Thus from (3.2), $\{P(x, \cdot)\}$ is strongly continuous.

4. *Results on weak continuity*

A weaker assumption on $\{P(x, \cdot)\}$ than strong continuity is *weak continuity*: this means that $\int P(x, dy)\, g(y)$ is a continuous function on $\mathscr{X}$ only for all *continuous* bounded functions g. It demands that, whenever $x_\alpha \to x$ in $\mathscr{T}$,

$$\int_{\mathscr{X}} P(x_\alpha, dy)\, g(y) \to \int_{\mathscr{X}} P(x, dy)\, g(y)$$

whenever $g \in \mathbf{C}(\mathscr{X})$; that is, the map $g \to \int P(\cdot, dy)\, g(y)$ takes continuous bounded functions to continuous bounded functions.

Examining the proof of Proposition 3.1 shows that it continues to hold with weak in place of strong continuity: however, weak continuity is not enough to carry through the remainder of the arguments of §3, and, as the following example shows, we need some extra assumptions if we are to conclude even that compact sets are R-sets.

Example 2. Let $\mathscr{X}$ be any state space, with $\{X_n\}$ on $\mathscr{X}$ a chain with convergence norm $R^{-1} < 1$. Endow $\mathscr{X}$ with the trivial topology, whose only open sets are $\varnothing$ and $\mathscr{X}$. Then for any pair (x, y) the sequence $(x, x, \ldots)$ tends to y, and so the only continuous functions are constant on $\mathscr{X}$. Hence, since $P(x, \mathscr{X}) = 1$ for all x, $\{P(x, \cdot)\}$ is weakly continuous. However, $\mathscr{X}$ itself is a compact set, and $\sum P^n(x, \mathscr{X}) = \infty$, so $\mathscr{X}$ is not an R-set. Note that $\{P(x, \cdot)\}$ is *not* strongly continuous, since for each pair (x, y) this would require the sequence $\big(P(x, A), P(x, A), \ldots\big)$ to approach $P(y, A)$, for every A: this can only happen if, for some probability measure π,

$$P(x, A) = P(y, A) \equiv \pi(A),$$

which in turn means that π is a 1-invariant probability measure for $\{X_n\}$, and hence that $\{X_n\}$ is 1-positive. This contradicts $R > 1$.

In order to make weak continuity a useful assumption, therefore, we need the space $\mathscr{X}$ and the σ-field $\mathscr{F}$ to be considerably more compatible with the topology $\mathscr{T}$. In particular, we need the space $\mathbf{C}(\mathscr{X})$ to be rather richer than in Example 2.

We need the notation

$$\mathbf{C}^+(\mathscr{X}) = \{g \in C(\mathscr{X}) : g(x) \geqslant 0 \text{ and } M\{y : g(y) > 0\} > 0\};$$
$$\mathbf{C}^+(A) = \{g \in \mathbf{C}^+(\mathscr{X}) : g(y) = 0,\ y \notin A\},\ A \in \mathscr{F}^+,$$

for the set of non-negative elements of $\mathbf{C}(\mathscr{X})$ which are not almost surely zero, and the set of elements of $\mathbf{C}^+(\mathscr{X})$ with support in A.

The results which follow are based on the use of the results of §2, in the manner of §3. We begin by giving explicitly

PROPOSITION 4.1. *If $\{P(x, \cdot)\}$ is weakly continuous, so is $\{G_\theta(x, \cdot)\}$ for any $\theta < 1$.*

Using this we prove

THEOREM 2. *Suppose $\{P(x, \cdot)\}$ is weakly continuous, and let*

$$\mathscr{A} = \{A \in \mathscr{F}^+ : \mathbf{C}^+(A) \text{ is non-empty}\}.$$

(i) *Every $K \in \mathscr{K}$ is an R-set if $\mathscr{A}$ contains at least one element of finite μ-measure, where μ is R-subinvariant for $\{X_n\}$.*

(ii) *If $\{X_n\}$ is R-recurrent and aperiodic, then every $K \in \mathscr{K}$ is an L-set if* (i) *holds and $\mathscr{A}$ also contains at least one element A such that f is bounded away from zero on $A \setminus N_f$.*

Proof. Choose $A \in \mathscr{A}$ to satisfy (i). Then for $\theta < 1$, $g \in \mathbf{C}^+(A)$,

$$\left[\int_A G_\theta(x, dy)\, g(y)\right]$$

is positive for all x, and so bounded from zero on any compact set K. Choose $g \in \mathbf{C}^+(A)$ with $g(x) \leqslant 1$ for all x. Since μ is R-subinvariant, and A has finite μ-measure by assumption, for $\beta < R^{-1}$

$$\begin{aligned}
\infty &> [\beta/(1-\beta)]\, \mu(A) \\
&\geqslant [\beta/(1-\beta)] \int_A \mu(dy)\, g(y) \\
&\geqslant \int_{\mathscr{X}} \mu(dw) \left[\int_A G_{R\beta}(w, dy)\, g(y)\right] \\
&\geqslant \mu(K) \inf_{w \in K} \left[\int_A G_{R\beta}(w, dy)\, g(y)\right],
\end{aligned} \tag{4.1}$$

so that if $K \in \mathscr{K}$, it follows from (4.1) and the preceding remarks that $\mu(K) < \infty$, and so, from Proposition 2.1, K is an R-set.

Now suppose $\{X_n\}$ R-recurrent and aperiodic. If (i) holds, it must be for $\mu = Q$, the unique R-subinvariant measure, and the first part of the proof shows that $Q(K) < \infty$ for every $K \in \mathscr{K}$. We now prove, therefore, that f is bounded from zero on $K \setminus N_f$, $K \in \mathscr{K}$, and the desired result follows from Proposition 2.2, since $N_K \supseteq N_f$.

To prove this, choose $A \in \mathscr{A}$ satisfying (ii). For $x \notin N_f$, R-invariance of f gives, for any $\beta < R^{-1}$, and $g \in \mathbf{C}^+(A)$ with $g(x) \leqslant 1$,

$$\begin{aligned}
[\beta/(1-\beta)] f(x) &= \int_{\mathscr{X} \setminus N_f} G_{R\beta}(x, dy) f(y) \\
&\geqslant \left[\inf_{y \in A \setminus N_f} f(y)\right] G_{R\beta}(x, A) \\
&\geqslant \left[\inf_{y \in A \setminus N_f} f(y)\right] \left[\int_A G_{R\beta}(x, dy)\, g(y)\right].
\end{aligned} \tag{4.2}$$

The first term on the right in (4.2) is bounded from zero by assumption, and the second is bounded from zero for x in compact K by weak continuity: hence $f(x)$ is bounded from zero on $K \setminus N_f$, and the result follows.

This theorem makes explicit just how rich in continuous functions the topology needs to be to allow our methods to work. Note that Example 2 violates (i), because it has only constant continuous functions; if one of these satisfied (i), it would mean $\mu(\mathscr{X}) < \infty$, which would in turn imply both $R = 1$ and $\{X_n\}$ 1-positive (*cf.* [5; §4]).

Various other criteria for elements of $\mathscr{K}$ to be R-sets could be formulated in a similar vein: for example, it suffices that there exists $g \in \mathbf{C}^+(\mathscr{X})$ such that

$$\int \mu(dy)\, g(y) < \infty$$

for some R-subinvariant μ, as can be seen from the first part of the proof of the theorem.

Using Theorem 2, we can now find a second set of conditions which will imply that sets in $\mathscr{K}$ have the desired properties under the weak continuity hypothesis. Let $\mathbf{B}$ denote the Baire σ-field on $\mathscr{X}$ (that is, the smallest σ-field containing $g^{-1}(U)$ for every U in the Borel σ-field of the real line and every real-valued continuous function on $\mathscr{X}$). The σ-field $\mathbf{B}$ is generated by the class of *zero sets* of $\mathscr{X}$: that is, by the closed sets of the form $g^{-1}(0)$ for some $g \in \mathbf{C}(\mathscr{X})$. We denote by $\mathscr{B}$ the class of those open Baire sets in $\mathscr{F}^+$ which are complements of zero sets.

THEOREM 3. *Suppose $\{P(x, \cdot)\}$ is weakly continuous, and that $\mathscr{F} \supseteq \mathbf{B}$. Then*

(i) *every $K \in \mathscr{K}$ is an R-set if $\mathscr{B}$ contains a set of finite μ-measure, where μ is an R-subinvariant measure for $\{X_n\}$; and*

(ii) *if $\{X_n\}$ is R-recurrent and aperiodic, every $K \in \mathscr{K}$ is an L-set if* (i) *holds, and $\mathscr{B}$ also contains an element on which $f(x)$ is bounded from zero.*

Proof. Since M is totally finite, any $B \in \mathscr{B}$ can be inner approximated by a zero set B' in $\mathbf{B}$ so that $M(B') > M(B) - \varepsilon$, [7; p. 171]. Further, there exists an element of $\mathbf{C}(\mathscr{X})$ which is zero on B^c and unity on B' [7; p. 168]. This suffices to show that $\mathbf{C}^+(B) \neq \varnothing$; that is, $\mathscr{B} \subseteq \mathscr{A}$ as defined in Theorem 2. The theorem then follows from Theorem 2.

The theorems of this section are not as satisfactory as Theorem 1, since they depend on being able to ascertain some at least of the sets on which R-subinvariant measures and functions behave well.* The difficulty with the weak continuity condition is that, whilst it seems exactly right to establish the vital Proposition 4.1, which leads to the desired lower bound on the right-hand side of (4.1), it fails, without some extra conditions, to give any sort of upper bound on the left-hand side of (4.1); and similarly it fails to give a lower bound on the right-hand side of (4.2) without some extra conditions.

It should be noted that in the probabilistically interesting case $R = 1$, we have $f(x) \equiv 1$ and so either Theorem 2(i) or Theorem 3(i) is sufficient to ensure that $K \in \mathscr{K}$ is both an R-set and an L-set. If $R > 1$ and $\{X_n\}$ is R-positive, on the other hand, we have, [5; Theorem 7], that $\int Q(dx) f(x) < \infty$, and hence f cannot be bounded from zero on $\mathscr{X}$ (since $Q(\mathscr{X}) < \infty$ violates $R > 1$, as we have seen before). Hence we do need the second condition of our Theorems to use Proposition 2.2 and ensure that sets in $\mathscr{K}$ are L-sets.

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* *Added in proof:* In a sequel, we improve this situation considerably.

5. *A weakly continuous example*

Finally, we give the following example, which is both a non-trivial case where weak continuity holds but strong continuity does not, and also an illustration that weak continuity can, with no explicit assumptions on $(\mathscr{X}, \mathscr{T})$ other than that it can support some strongly continuous chain (a condition violated by Example 2), be sufficient for compact sets to be both R-sets and L-sets. This shows that strong continuity is not necessary for either of these to be possible.

Example 3. Let $\{Y_n\}$ be a Markov chain satisfying Condition I with transition probabilities $Q(x, A)$, such that $\{Q(x, \cdot)\}$ is strongly continuous, and with convergence norm $R_Q^{-1} < 1$. Define the chain $\{X_n\}$ by the transition probabilities

$$P(x, A) = \alpha Q(x, A) + \beta\delta(x, A), \qquad \alpha, \beta > 0, \qquad \alpha+\beta = 1.$$

It can be verified inductively that

$$P^n(x, A) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} Q^k(x, A), \qquad n = 0, 1, \ldots$$

and so

$$\delta(x, A) + G_r(x, A) = \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} Q^k(x, A)$$

$$= \sum_{k=0}^{\infty} (\alpha r)^k Q^k(x, A) \sum_{n=k}^{\infty} \binom{n}{k} (\beta r)^{n-k}$$

$$= \sum_{k=0}^{\infty} (\alpha r)^k Q^k(x, A) \left[\sum_{m=0}^{\infty} \binom{m+k}{k} (\beta r)^m\right].$$

Now using

$$\binom{m+k}{k} = (-1)^m \binom{-k-1}{m}, \qquad k, m = 0, 1, 2, \ldots,$$

we obtain for $|\beta r| < 1$

$$G_r(x, A) + \delta(x, A) = \sum_{k=0}^{\infty} (\alpha r)^k Q^k(x, A) [1-\beta r]^{-k-1}$$

$$= [1-\beta r]^{-1} H_{(\alpha r/[1-\beta r])}(x, A) \tag{5.1}$$

where

$$H_z(x, A) = \sum_0^{\infty} Q^k (x, A) z^k.$$

From (5.1) it is clear that $\{X_n\}$ satisfies Condition I when $\{Y_n\}$ does. The binomial expansion used to get (5.1) is only valid if $|\beta r| < 1$; however, since

$$P^{n+m}(x, A) \geqslant \beta^n \alpha^m Q^m(x, A),$$

we have

$$G_r(x, A) \geqslant \sum_{n=1}^{\infty} (\beta r)^n \alpha^m Q^m(x, A) r^m$$

and if $M(A) > 0$ the radius of convergence R of $G_r(x, A)$ is no more than β^{-1}. Hence to find R explicitly we only need consider r such that $|\beta r| < 1$, and so we can use (5.1).

It follows immediately from (5.1) that

$$R = \sup\{r : \alpha r/(1-\beta r) \leqslant R_Q\}.$$

But if $r\beta < 1$, $\alpha r/(1-\beta r) < R_Q$ is equivalent to $r < R_Q/(\alpha+\beta R_Q)$. Hence

$$R = \min(R_Q/(\alpha+\beta R_Q), \beta^{-1}).$$

However, $\alpha > 0$, and so $\beta R_Q < \alpha+\beta R_Q$, so we find

$$R = R_Q/(\alpha+\beta R_Q). \tag{5.2}$$

Notice that this implies that $R < R_Q$.

From (5.1) and (5.2) we deduce immediately that if A is an R_Q-set for $\{Y_n\}$ then A is an R-set for $\{X_n\}$, and so the strong continuity of $\{Q(x, \cdot)\}$ ensures from Theorem 1 that all compact $K \in \mathscr{K}$ are R-sets. Also if $\{Y_n\}$ is R_Q-recurrent, $\{X_n\}$ is R-recurrent, and conversely. Suppose $\{Y_n\}$ is R_Q-recurrent and aperiodic, and let f_Q be the unique R_Q-invariant function for $\{Y_n\}$: we wish to find the unique R-invariant function for $\{X_n\}$ in order to use Proposition 2.3. This is the solution to

$$\begin{aligned} f(x) &= R \int_{\mathscr{X}} P(x, dy) f(y) \\ &= \frac{R_Q}{\alpha+\beta R_Q}\left[\int_{\mathscr{X}} \alpha Q(x, dy) f(y) + \beta f(x)\right]; \end{aligned}$$

rearranging this gives

$$f(x) = R_Q \int_{\mathscr{X}} Q(x, dy) f(y),$$

and so by uniqueness $f \equiv f_Q$. Similarly, the unique R-invariant measure for $\{X_n\}$ is identical with the unique R_Q-invariant measure for $\{Y_n\}$. Using the strong continuity of $\{Q(x, \cdot)\}$ and this identification, Propositions 3.2 and 3.3 imply that any $K \in \mathscr{K}$ satisfies (2.4), and hence is an L-set for $\{X_n\}$; and in fact for every A satisfying (2.4), and for almost every X,

$$\lim_{n\to\infty} R^n P^n(x, A) = \lim_{n\to\infty} R_Q{}^n Q^n(x, A)$$

from Theorem C.

Finally, the chain $\{X_n\}$ has $\{P(x, \cdot)\}$ weakly continuous but not strongly continuous; for if $g \in \mathbf{D}(\mathscr{X})$ and $x_\alpha \to x$

$$\int P(x_\alpha, dy)\, g(y) = \alpha \int Q(x_\alpha, dy)\, g(y) + \beta g(x_\alpha)$$

and although the first term tends to $\alpha \int Q(x, dy)\, g(y)$ from the strong continuity of $\{Q(x, \cdot)\}$, the second tends to $\beta g(x)$ if and only if x is a continuity point of g.

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