Fall 2022

S&DS 241: Probability Theory with Applications

Homework 10 (LAST!)

Due: Dec 7, 2022 6pm ET via Gradescope

`Prof. Yihong Wu`

1. (15pts) Blitzstein-Hwang 2nd Ed Chapter 9, Problem 25

2. (15pts) Blitzstein-Hwang 2nd Ed Chapter 9, Problem 14, part (b) only.[1]

3. (20pts) Roll a fair die for n times. Observing X, the number of ⚀, we want to estimate Y, the number of ⚅.

   (a) By computing the conditional PMF, show that the conditioned on $X = x$, Y is distributed as $\mathrm{Bin}(n - x, \frac{1}{5})$.

   (b) Find the best estimate of Y as a function X. In this case is there any gain over the best linear estimate we derived in Lecture 22?

4. (25pts) *Chernoff bound.* Let $X \sim N(0, 1)$ be a standard normal random variable. Recall the MGF of X is given by $M_X(t) = e^{t^2/2}$ from Lecture 23. Let $a > 0$.

   (a) Show that for any $t > 0$,

   $$P(X > a) \le e^{-ta} M_X(t) = e^{-ta + t^2/2}.$$

   (Hint: apply Markov inequality to the random variable e^{tX}.)

   (b) Optimizing over $t > 0$ to get the best bound, show that

   $$P(X > a) \le e^{-a^2/2}$$

   (c) Evaluate the bound for $a = 5$ and compare with the actual value $P(X > 5) = 1 - \Phi(5) = 2.9 \times 10^{-7}$ as well as what Chebyshev inequality yields.

5. (25pts) Alice transmits a signal $Y \sim N(0, 1)$ wirelessly to Bob, who received a noisy version $X = Y + Z$. Here the noise Z is also distributed as $N(0, 1)$ and independent of Y. Bob decides to estimate Alice's signal based on what he received.

   (a) What is the distribution of X? Find its PDF $f_X(x)$.

   (b) Bob first tries to use the best linear estimate of Y given X. Find the estimator and its mean-squared error (MSE).

   (c) Bob now wonders if he can do better with non-linear estimates. Show that the conditional distribution of Y given $X = x$ is $N(x/2, 1/2)$ by finding the conditional PDF $f_{Y|X}(y|x)$. Find the best estimate of Y given X and its MSE. Does it improve the best linear estimate? (Hint: to find the joint PDF (X, Y), note that it is a linear a linear transformation of (Y, Z).)

[1] According to *The Hitchhiker's Guide to the Galaxy*, this is the answer to the Ultimate Question of Life, the Universe, and Everything.