S&DS 241 Lecture 2

Probability axioms. Counting.

B-H Section 1.2-1.6

(App A.1: review of set theory)

Axiomatic framework

Three elements

1. Sample space
2. Events
3. Probability measure

Sample space and events

- Sample space Ω: set of all possible outcomes
- Outcome $\omega \in \Omega$: element of sample space
- Events $A \subset \Omega$: subset of sample space

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- Events: $A = \{H\}$

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- $\Omega = \{H, T\}$
- Events: $A = \{H\}$

Die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Events:
  - $A = \{6\}$
  - $A = \{\text{even number}\} = \{2, 4, 6\}$

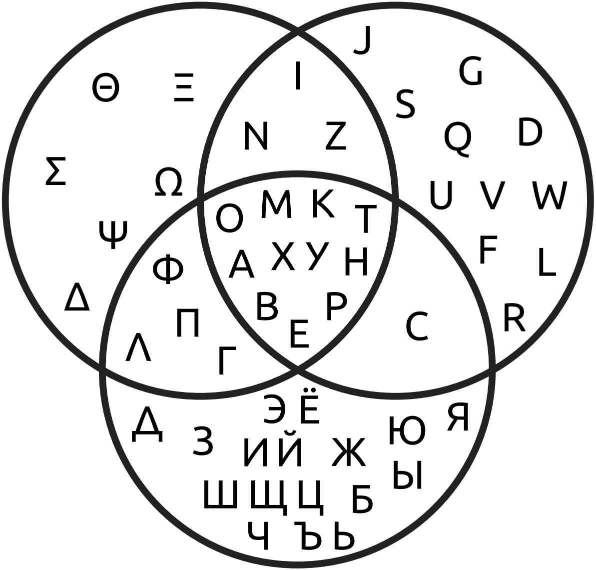
Language of set theory

Name	Notation	Description
complement	A^c	occurs if and only if A does not occur
union	$A \cup B$	occurs when either A or B occurs
intersection	$A \cap B$	occurs when both A and B occurs
implication	$A \subset B$	B occurs whenever A occurs.

Examples: dice

1. $\{\text{even}\}^c = \{\text{odd}\}$
2. $\{\text{even}\} \cup \{\text{divisible by } 3\} = \{2, 3, 4, 6\}$
3. $\{\text{even}\} \cap \{\text{divisible by } 3\} = \{6\}$
4. $\{\text{divisible by } 6\} \subset \{\text{even}\}$

Useful tool: Venn diagram



Useful tool: de Morgan's law

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Probability of an event

To each event $A \subset \Omega$, we attach a number $P(A)$, which denotes its probability.

Simplest model: equiprobable

In a finite sample space Ω:

1. For event A,

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{|A|}{|\Omega|}$$

2. Hence every outcome is equally likely:

$$P(\{\omega\}) = \frac{1}{|\Omega|}$$

This is sometimes referred to as the classical definition of probability.[1]

[1] For more: https://en.wikipedia.org/wiki/Classical_definition_of_probability.

Examples

Fair coin

- $\Omega = \{H, T\}$
- $P(\{H\}) = P(\{T\}) = \frac{1}{2}$

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- $P(\{6\}) = \frac{1}{6}$
- $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$.
- How many possible events are there?

Two fair dice

- Sample space

$$\Omega = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

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- $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$

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- $P(\text{sum is even}) = \frac{2+4+6+4+2}{36} = \frac{1}{2}$ (a better way? does the result change if the die has five faces?)

What if the dice are not fair?

More generally

Definition

Let Ω be a finite sample space. A **probability measure** is a function P that assigns a number to each event that satisfies the following properties:

1. Positivity:[2]
$$P(\{\omega\}) \geq 0, \quad \forall \omega \in \Omega$$
2. Normalization:
$$P(\Omega) = 1$$
3. Additivity:
$$P(A) = \sum_{\omega \in A} P(\{\omega\})$$

[2] The notation $\forall$ reads "for all".

Examples

Biased coin

- $\Omega = \{H, T\}$
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Loaded die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$

ω	1	2	3	4	5	6
$P(\{\omega\})$	0.1	0.1	0.1	0.1	0.1	0.5

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- $P(\text{even}) = 0.7$

Remarks

- For finite sample space, suffices to specify the prob of each outcome
- Probability of a given event depends on the model (assumption)
- Equiprobable model is often assumed implicitly

Probability axioms

More general experiments:

- Infinitely many outcomes: e.g. how many tosses to get the first head
- A continuum of possible outcomes: e.g. position of a randomly spinned wheel

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- $P(\text{rain}) + P(\text{no rain}) = 1$
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The axiomatic framework of probability is proposed by A.N. Kolmogorov.

Probability axioms

Definition

A probability measure is a function P that assigns a number to each event that satisfies the following properties:

1. Positivity: For any event A,
$$P(A) \geq 0,$$
2. Normalization:
$$P(\Omega) = 1$$
3. Additivity: Let A_i be mutually exclusive events, i.e., $A_i \cap A_j = \varnothing$ whenever $i \neq j$. Then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Consequences

Corollary

- $P(\varnothing) = 0$
- $A \subset B \implies P(A) \leq P(B)$: *if A leads to B, then B is more likely*
- $0 \leq P(A) \leq 1$
- $P(A^c) = 1 - P(A)$

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Proof.

- $\varnothing \cap \varnothing = \varnothing$ and $\varnothing \cup \varnothing = \varnothing$.
- $P(B) = P(A) + P(B \cap A^c)$
- $\varnothing \subset A \subset \Omega$
- A and A^c are mutually exclusive

$\square$

Tversky-Kahneman experiment in the 80s[3]

A group of subjects are told:

> *Linda is 31, single, outspoken, and very bright. She majored in philosophy in college. As a student, she was deeply concerned with racial discrimination and other social issues, and participated in anti-nuclear demonstrations.*

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Of course not: $P(A \cap B) \leq P(B)$ (They call this phenomenon the conjunction fallacy, and note that it appears to be unaffected by prior training in probability or statistics.)

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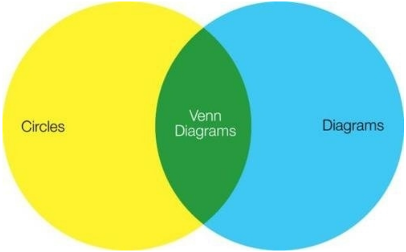
Union of two events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Analogy: “area”



Proof.

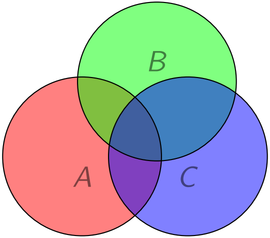
$A \cup B = A \cup (B \cap A^c)$, $B = (A \cap B) \cup (B \cap A^c)$. $\square$

Three events

$$
\begin{aligned}
P(A \cup B \cup C) = P(A) + P(B) + P(C) \\
- P(A \cap B) - P(B \cap C) - P(C \cap A) \\
+ P(A \cap B \cap C)
\end{aligned}
$$

Proof.

Apply the previous formula for two events thrice. $\square$



Extending this formula to $P(\cup_{i=1}^{n} A_i)$: Inclusion-exclusion principle (later).

Counting and computing probabilities

Recall: factorial and binomial coefficients

- $n! \triangleq n(n-1)\cdots 1$: number of ways to permute n items
- $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$: number of ways, <u>considering order</u>, to choose k items out of n items
- $\binom{n}{k} \triangleq \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$: number of ways, <u>regardless of order</u>, to choose k items out of n items

Example: Poker

A poker deck consists of 52 cards:

$$1\diamondsuit, 1\spadesuit, 1\heartsuit, 1\clubsuit, \ldots, 13\diamondsuit, 13\spadesuit, 13\heartsuit, 13\clubsuit$$

Draw five cards successively from the deck:

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- Full house: three cards have the same rank, and the other two have another rank e.g. $3\diamondsuit, 3\spadesuit, 3\heartsuit, 9\diamondsuit, 9\clubsuit$

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$$P(\text{Full house}) = \frac{13 \times 12 \times \binom{4}{3} \times \binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165} \approx 0.144\%$$

Example: More poker

A poker deck consists of 52 cards:

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Draw two cards successively from the deck.

$P(\text{2nd card is higher than 1st card}) =$

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Draw two cards successively from the deck.

$$P(\text{2nd card is higher than 1st card}) = \frac{\sum_{i=1}^{13} 4 \times 4 \times (i-1)}{52 \times 51} = \frac{8}{17}$$

since

$$1 + 2 + \cdots + 12 = \frac{(1+12) \times 12}{2} = 78$$

Example: More poker

Or, we can be smarter. Consider three events:

$$\begin{aligned} H &= \{\text{2nd higher than 1st}\} \\ L &= \{\text{2nd lower than 1st}\} \\ E &= \{\text{2nd equal to 1st}\} \end{aligned}$$

Then we know

1. $P(H) + P(L) + P(E) = 1$

Example: More poker

Or, we can be smarter. Consider three events:

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4. Now we profit:
$$P(H) = \frac{1 - P(E)}{2} = \frac{8}{17}$$