S&DS 241 Lecture 5

Random variable. Expectation. Probability mass functions.

B-H: 3.1, 3.2, 4.1, 4.2, 4.4, 4.5

Random variable

Definition

A random variable X is a real-valued function defined on sample space:
$$X : \Omega \to \mathbb{R}.$$

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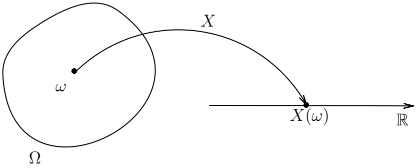

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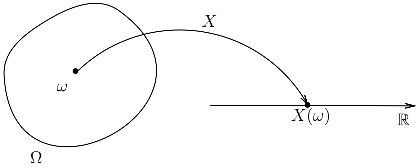

- Let us focus on discrete sample space: either finite or countably infinite many outcomes.
- Then X takes finitely or countably infinite many values. We say such X is a discrete random variable.

Examples: Two fair dice

- Sample space

$$\Omega = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

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Expectation

The expectation (aka expected value or mean) of a random variable X is defined by

$$E(X) \triangleq \sum_{\omega \in \Omega} P(\{\omega\})\, X(\omega)$$

Examples

- Let X be the outcome of a fair die. Then

$$E(X) = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

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- Let Y be the max of two fair dice. Then

$$E(Y) = \frac{1}{36} \sum_{\omega_1=1}^{6} \sum_{\omega_2=1}^{6} \max\{\omega_1, \omega_2\} = \frac{161}{36} \approx 4.5 > E(X)$$

Linearity of expectation

For any random variable X and Y:

$$E(X + Y) = E(X) + E(Y)$$

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Proof.

$$\begin{aligned} E(X+Y) &= \sum_{\omega\in\Omega}(X(\omega)+Y(\omega))P(\{\omega\}) \\ &= \underbrace{\sum_{\omega\in\Omega}X(\omega)P(\{\omega\})}_{E(X)} + \underbrace{\sum_{\omega\in\Omega}Y(\omega)P(\{\omega\})}_{E(Y)} \end{aligned}$$

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Corollary:

$$E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n)$$

This is an extremely useful fact!

Linearity of expectation

Similarly, for any constants $a, b \in \mathbb{R}$

$$E(aX) = aE(X)$$

and more generally

$$E(aX + bY) = aE(X) + bE(Y),$$

that is

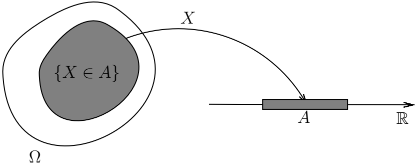
expectation of linear combination = linear combination of expectations

More convenient way to describe the random variable

Events associated with random variables

Notations

- $\{X \in A\} \triangleq \{\omega : X(\omega) \in A\} = X^{-1}(A)$



- For example
  - $\{X = 1\} = \{\omega : X(\omega) = 1\}$
  - $\{X \leq 2\} = \{\omega : X(\omega) \leq 2\}$

Probability Mass Function (PMF)

Let X be a discrete random variable taking values in a set $\mathcal{X}$. The probability mass function of X is a function $p_X : \mathcal{X} \to [0, 1]$, defined by

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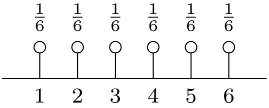
$$p_X(x) = \frac{1}{6}, \quad x = 1, \ldots, 6$$



x	1	2	3	4	5	6
$p_X(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Example of PMF

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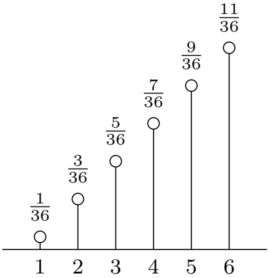

y	1	2	3	4	5	6
$p_Y(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

The important of PMF

- Oftentimes we don't need to spell out the sample space precisely when studying a random variable.
- All we need is its PMF which completely describes the distribution of a discrete random variable.

Using PMF to compute probability of events

For example

- $$P(X \le 5) = \sum_{x \le 5} p_X(x)$$

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- $$P(X \leq 5) = \sum_{x \leq 5} p_X(x)$$

- $$P(X \geq 3 | X \leq 5) = \frac{P(3 \leq X \leq 5)}{P(X \leq 5)} = \frac{\sum_{3 \leq x \leq 5} p_X(x)}{\sum_{x \leq 5} p_X(x)}$$

Expectation via PMF

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Proof.

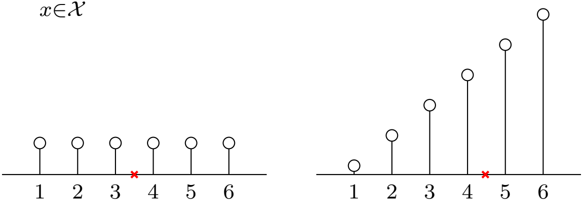
Group the sum according to the value of X:

$$\begin{aligned} E(X) &= \sum_{\omega \in \Omega} P(\{\omega\}) X(\omega) \\ &= \sum_{x \in \mathcal{X}} \sum_{\omega \in \Omega : X(\omega) = x} P(\{\omega\}) x \\ &= \sum_{x \in \mathcal{X}} \underbrace{\left(\sum_{\omega \in \Omega : X(\omega) = x} P(\{\omega\}) \right)}_{P(X=x)} x \end{aligned}$$

□

Interpretation of expectation

$$E(X) = \sum_{x \in \mathcal{X}} x p_X(x) = \text{average of values weighted by PMF}$$



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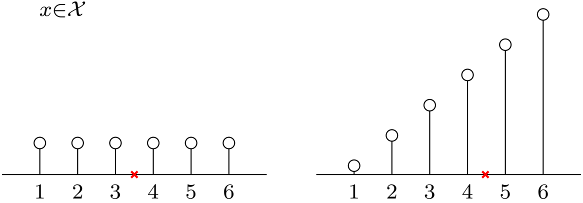

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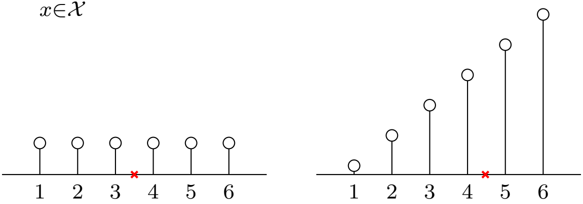

- Expected value $\neq$ typical value!
- How close a random variable is to its expectation depends on many things, e.g., variance (later)
- Law of large numbers: average of many independent samples $\approx$ expectation

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- PMF of X:

x	4	3	2	1	0
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- Expectation: $E(X) = 4 \times \frac{1}{24} + 2 \times \frac{6}{24} + 1 \times \frac{8}{24} = 1$
- Coincidence? What if there are 100 guests?

Indicator random variable

- Indicator of event A:

$$\mathbf{1}_A = \begin{cases} 1 & A \text{ occurs} \\ 0 & A \text{ does not occur} \end{cases}$$

(that is, $\mathbf{1}_A : \Omega \to \{0,1\}$ is a function s.t. $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise)

- Important relation ("fundamental bridge" in B-H §4.4):

$$E(\mathbf{1}_A) = P(A)$$

Proof.

The random variable $\mathbf{1}_A$ only takes two values and $E(\mathbf{1}_A) = 1 \times P(A) + 0 \times P(A^c)$. □

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  Then $P(A_i) = 1/n$

- Define the corresponding indicator random variables:

$$X_i = \mathbf{1}_{A_i} = \begin{cases} 1 & i^{\text{th}} \text{ guest gets own hat} \\ 0 & i^{\text{th}} \text{ guest doesn't get own hat} \end{cases}$$

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  Then $P(A_i) = 1/n$

- Define the corresponding indicator random variables:

$$X_i = \mathbf{1}_{A_i} = \begin{cases} 1 & i^{\text{th}} \text{ guest gets own hat} \\ 0 & i^{\text{th}} \text{ guest doesn't get own hat} \end{cases}$$

- Note $X = X_1 + \cdots + X_n$

Example: Matching

A hat-checker in a restaurant, having checked n hats, gets them hopelessly scrambled and returns them at random to the guests as they leave. Let $X =$ number of guests with own hats. Find $E(X)$ without finding PMF.

- Define the event

$$A_i = \{i^{\text{th}} \text{ guest gets own hat}\}, \quad i = 1, \ldots, n$$

  Then $P(A_i) = 1/n$

- Define the corresponding indicator random variables:

$$X_i = \mathbf{1}_{A_i} = \begin{cases} 1 & i^{\text{th}} \text{ guest gets own hat} \\ 0 & i^{\text{th}} \text{ guest doesn't get own hat} \end{cases}$$

- Note $X = X_1 + \cdots + X_n$ and hence

$$E(X) = E(X_1) + \cdots + E(X_n) = P(A_1) + \cdots + P(A_n) = \frac{1}{n} + \cdots + \frac{1}{n} = 1$$

Example: Matching

This is precisely the rigorous version of the intuition:

> *"There are n people. Each person has a chance of $1/n$ getting own hat. So the average number of people getting own hat is one."*