S&DS 241 Lecture 5

B-H: 3.1, 3.2, 4.1, 4.2, 4.4, 4.5
A random variable $X$ is a real-valued function defined on sample space:

$$X : \Omega \rightarrow \mathbb{R}.$$
Random variable

Definition
A random variable $X$ is a real-valued \textbf{function} defined on sample space: $X : \Omega \rightarrow \mathbb{R}$.

- Each outcome $\omega$ gives rise to a realization $X(\omega)$ of the random variable.
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- Each outcome $\omega$ gives rise to a realization $X(\omega)$ of the random variable.

- Let us focus on discrete sample space: either finite or countably infinite many outcomes.

- Then $X$ takes finitely or countably infinite many values. We say such $X$ is a discrete random variable.
Examples: Two fair dice

• Sample space

\[
\Omega = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
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(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\]

• Outcome \( \omega = (\omega_1, \omega_2) \)

• Random variables:

  ▶ First die: \( X(\omega) = \omega_1 \) takes values in \{1, 2, \ldots, 6\} equally likely.
  ▶ Best of two: \( Y(\omega) = \max\{\omega_1, \omega_2\} \) takes values in \{1, 2, \ldots, 6\}. 
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- **Outcome** \(\omega = (\omega_1, \omega_2)\)

- **Random variables:**
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Expectation

The expectation (aka expected value or mean) of a random variable $X$ is defined by

$$E(X) \triangleq \sum_{\omega \in \Omega} P(\{\omega\}) X(\omega)$$

Examples

- Let $X$ be the outcome of a fair die. Then

$$E(X) = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$
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Examples

- Let $X$ be the outcome of a fair die. Then
  $$E(X) = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

- Let $Y$ be the max of two fair dice. Then
  $$E(Y) = \frac{1}{36} \sum_{\omega_1=1}^{6} \sum_{\omega_2=1}^{6} \max\{\omega_1, \omega_2\} = \frac{161}{36} \approx 4.5 > E(X)$$
Linearity of expectation

For any random variable $X$ and $Y$:

$$E(X + Y) = E(X) + E(Y)$$
Linearity of expectation

For any random variable $X$ and $Y$:

$$E(X + Y) = E(X) + E(Y)$$

Proof.

$$E(X + Y) = \sum_{\omega \in \Omega} (X(\omega) + Y(\omega))P(\{\omega\})$$

$$= \sum_{\omega \in \Omega} X(\omega)P(\{\omega\}) + \sum_{\omega \in \Omega} Y(\omega)P(\{\omega\})$$

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$$= E(X) + E(Y)$$

Corollary:

$$E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n)$$

This is an extremely useful fact!
Linearity of expectation

Similarly, for any constants \( a, b \in \mathbb{R} \)

\[
E(aX) = aE(X)
\]

and more generally

\[
E(aX + bY) = aE(X) + bE(Y),
\]

that is

"expectation of linear combination = linear combination of expectations"
More convenient way to describe the random variable
Events associated with random variables

Notations

- \( \{ X \in A \} \triangleq \{ \omega : X(\omega) \in A \} = X^{-1}(A) \)

For example

- \( \{ X = 1 \} = \{ \omega : X(\omega) = 1 \} \)
- \( \{ X \leq 2 \} = \{ \omega : X(\omega) \leq 2 \} \)
Probability Mass Function (PMF)

Let $X$ be a discrete random variable taking values in a set $\mathcal{X}$. The probability mass function of $X$ is a function $p_X : \mathcal{X} \to [0, 1]$, defined by

$$p_X(x) \triangleq P(X = x) = P(\{\omega : X(\omega) = x\}), \quad x \in \mathcal{X}$$
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**Examples:**

- Let $X$ be the outcome of a fair die. Then

  $$p_X(x) = \frac{1}{6}, \quad x = 1, \ldots, 6$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
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<th>6</th>
</tr>
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Example of PMF

Let $Y$ be the max of two fair dice.

- Sample space:

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- PMF:

<table>
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<tr>
<td>$p_{Y}(y)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{7}{36}$</td>
<td>$\frac{9}{36}$</td>
<td>$\frac{11}{36}$</td>
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</table>
The importance of PMF

- Oftentimes we don’t need to spell out the sample space precisely when studying a random variable.
- All we need is its PMF which completely describes the distribution of a discrete random variable.
Using PMF to compute probability of events

For example

\[
P(X \leq 5) = \sum_{x \leq 5} p_X(x)
\]
Using PMF to compute probability of events

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\[ P(X \leq 5) = \sum_{x \leq 5} p_X(x) \]

\[ P(X \geq 3 | X \leq 5) = \frac{P(3 \leq X \leq 5)}{P(X \leq 5)} = \frac{\sum_{3 \leq x \leq 5} p_X(x)}{\sum_{x \leq 5} p_X(x)} \]
Expectation via PMF

\[ E(X) = \sum_{x \in X} x p_X(x) \]
Expectation via PMF

\[ E(X) = \sum_{x \in \mathcal{X}} x P_X(x) \]

**Proof.**

Group the sum according to the value of \( X \):

\[ E(X) = \sum_{\omega \in \Omega} P(\{\omega\}) X(\omega) \]

\[ = \sum_{x \in \mathcal{X}} \sum_{\omega \in \Omega: X(\omega) = x} P(\{\omega\}) x \]

\[ = \sum_{x \in \mathcal{X}} \left( \sum_{\omega \in \Omega: X(\omega) = x} P(\{\omega\}) \right) x \]

\[ = \sum_{x \in \mathcal{X}} \left( \sum_{\omega \in \Omega: X(\omega) = x} P(\{\omega\}) \right) \frac{x}{P(X = x)} \]
Interpretation of expectation

\[ E(X) = \sum_{x \in X} xp_X(x) = \text{average of values weighted by PMF} \]
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- Expected value \( \neq \) typical value!
- How close a random variable is to its expectation depends on many things, e.g., variance (later)
- Law of large numbers: average of many independent samples \( \approx \) expectation
Example: Matching

A hat-checker in a restaurant, having checked four hats, gets them hopelessly scrambled and returns them at random to the guests as they leave. Let $X =$ number of guests with own hats. Find $p_X$ and $E(X)$. 

$\Omega = \{\text{permutations on four items}\}$, $|\Omega| = 4! = 4 \times 3 \times 2 \times 1 = 24$

- $X$ takes values 0, 1, 2, 3, 4
- $|\{X = 4\}| = 1$
- $|\{X = 3\}| = 0$
- $|\{X = 2\}| = 4\cdot2 = 8$
- $|\{X = 1\}| = 4\cdot2 = 8$
- $|\{X = 0\}| = 24 - 1 - 8 - 6 = 9$

PMF of $X$:

\[
p_X(x) = \frac{1}{24}, \frac{2}{24}, \frac{4}{24}, \frac{8}{24}, \frac{9}{24}
\]

Expectation:

\[
E(X) = 4\times\frac{1}{24} + 2\times\frac{2}{24} + 1\times\frac{8}{24} = 1
\]

Coincidence? What if there are 100 guests?
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• Coincidence? What if there are 100 guests?
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\[\begin{array}{c|ccccc}
X & 0 & 1 & 2 & 3 & 4 \\
\hline
p_X & \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{24} \\
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- PMF of $X$:

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Coincidence? What if there are 100 guests?
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- PMF of $X$:
  $\begin{array}{c|c|c|c|c|c}
  x & 4 & 3 & 2 & 1 & 0 \\
  p_X(x) & \frac{1}{24} & 0 & \frac{6}{24} & \frac{8}{24} & \frac{9}{24} \\
  \end{array}$
- Expectation: $E(X) = 4 \times \frac{1}{24} + 2 \times \frac{6}{24} + 1 \times \frac{8}{24} = 1$
- Coincidence? What if there are 100 guests?
Indicator random variable

- **Indicator of event** $A$:

  $1_A = \begin{cases} 
  1 & \text{if } A \text{ occurs} \\
  0 & \text{if } A \text{ does not occur}
  \end{cases}$

  (that is, $1_A : \Omega \to \{0, 1\}$ is a function s.t. $1_A(\omega) = 1$ if $\omega \in A$ and $0$ otherwise)

- **Important relation** ("fundamental bridge" in B-H §4.4):

  $$E(1_A) = P(A)$$

**Proof.**

The random variable $1_A$ only takes two values and

$$E(1_A) = 1 \times P(A) + 0 \times P(A^c).$$
Example: Matching

A hat-checker in a restaurant, having checked $n$ hats, gets them hopelessly scrambled and returns them at random to the guests as they leave. Let $X =$ number of guests with own hats. Find $E(X)$ without finding PMF.
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- Define the event

\[
A_i = \{ \text{\( i \)th guest gets own hat} \}, \quad i = 1, \ldots, n
\]

Then \( P(A_i) = 1/n \)
Example: Matching

A hat-checker in a restaurant, having checked \( n \) hats, gets them hopelessly scrambled and returns them at random to the guests as they leave. Let \( X \) = number of guests with own hats. Find \( E(X) \) without finding PMF.

- Define the event

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A_i = \{ i^{th} \text{ guest gets own hat} \}, \quad i = 1, \ldots, n
\]

Then \( P(A_i) = 1/n \)

- Define the corresponding indicator random variables:

\[
X_i = 1_{A_i} = \begin{cases} 
1 & \text{\( i^{th} \) guest gets own hat} \\
0 & \text{\( i^{th} \) guest doesn’t get own hat} 
\end{cases}
\]
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- Note \( X = X_1 + \cdots + X_n \)
Example: Matching

A hat-checker in a restaurant, having checked $n$ hats, gets them hopelessly scrambled and returns them at random to the guests as they leave. Let $X =$ number of guests with own hats. Find $E(X)$ without finding PMF.

- Define the event

$$A_i = \{i^{\text{th}} \text{ guest gets own hat}\}, \quad i = 1, \ldots, n$$

Then $P(A_i) = 1/n$

- Define the corresponding indicator random variables:

$$X_i = 1_{A_i} = \begin{cases} 
1 & i^{\text{th}} \text{ guest gets own hat} \\
0 & i^{\text{th}} \text{ guest doesn’t get own hat} 
\end{cases}$$

- Note $X = X_1 + \cdots + X_n$ and hence

$$E(X) = E(X_1) + \cdots + E(X_n) = P(A_1) + \cdots + P(A_n) = \frac{1}{n} + \cdots + \frac{1}{n} = 1$$
Example: Matching

This is precisely the rigorous version of the intuition:

“There are $n$ people. Each person has a chance of $\frac{1}{n}$ getting own hat. So the average number of people getting own hat is one.”