S&DS 241 Lecture 6

Functions of random variables. LOTUS rule.

Independence of random variables. Conditional independence.

B-H: 2.5, 3.7, 3.8, 4.5

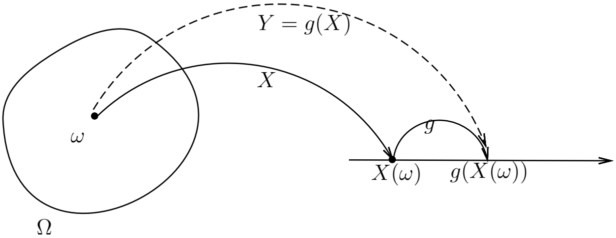
Functions of random variables

Let X be a discrete random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a function. Then $Y = g(X)$ is also a discrete random variable,

Functions of random variables

Let X be a discrete random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a function. Then $Y = g(X)$ is also a discrete random variable, because

- $Y : \Omega \to \mathbb{R}$ is the composition of $X : \Omega \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, which maps ω to $Y(\omega) = g(X(\omega))$



- the number of possible values of Y is at most that of X

Find PMF of $Y = g(X)$

If we know the PMF of X, we can obtain the PMF of Y:

Step 1 Find the set of values that Y takes

Step 2 Find the corresponding probabilities

$$p_Y(y) = P(g(X) = y) = \sum_{x:g(x)=y} p_X(x)$$

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $Y = 2X + 1$:

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $Y = 2X + 1$:

Step 1 Y takes values $-3, -1, 1, 3, 5$

Step 2 $P(Y = -3) = P(2X + 1 = -3) = P(X = -2) = 1/5$, and so on and so forth

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $Y = 2X + 1$:

Step 1 Y takes values $-3, -1, 1, 3, 5$

Step 2 $P(Y = -3) = P(2X + 1 = -3) = P(X = -2) = 1/5$, and so on and so forth

y	-3	-1	1	3	5
$p_Y(y)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $Z = X^2$:

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $Z = X^2$:

Step 1 Z takes values $0, 1, 4$

Step 2 $P(Z = 4) = P(X^2 = 4) = P(X = 2) + P(X = -2) = 2/5$

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $Z = X^2$:

Step 1 Z takes values $0, 1, 4$

Step 2 $P(Z = 4) = P(X^2 = 4) = P(X = 2) + P(X = -2) = 2/5$

z	0	1	4
$p_Z(z)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $W = -X$:

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $W = -X$:

w	-2	-1	0	1	2
$p_W(w)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Example

Given the PMF of X

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Find the PMF of $W = -X$:

w	-2	-1	0	1	2
$p_W(w)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

<u>Note</u>: X and W have the same distribution (PMF), but these two random variables are NOT the same! In fact, $P(X = -W) = 1$.

Compute $E(g(X))$

Compute $E(g(X))$

Example:

- PMF of X:

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

- PMF of $Z = X^2$:

z	0	1	4
$p_Z(z)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

Compute $E(g(X))$

Example:

- PMF of X:

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

- PMF of $Z = X^2$:

z	0	1	4
$p_Z(z)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

- Expectation:

$$E(Z) = \frac{1}{5} \times 0 + \frac{2}{5} \times 1 + \frac{2}{5} \times 4 = 2$$

Compute $E(g(X))$

Example:

- PMF of X:

x	-2	-1	0	1	2
$p_X(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

- PMF of $Z = X^2$:

z	0	1	4
$p_Z(z)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

- Expectation:

$$E(Z) = \frac{1}{5} \times 0 + \frac{2}{5} \times 1 + \frac{2}{5} \times 4 = 2$$

- Can we find $E(Z)$ without finding p_Z?

$$E(X^2) = \frac{1}{5} \times (-2)^2 + \frac{1}{5} \times (-1)^2 + \frac{1}{5} \times 0^2 + \frac{1}{5} \times 1^2 + \frac{1}{5} \times 2^2 = 2$$

Compare:

$$E(X) = \frac{1}{5} \times (-2) + \frac{1}{5} \times (-1) + \frac{1}{5} \times 0 + \frac{1}{5} \times 1 + \frac{1}{5} \times 2 = 0.$$

Law Of The Unconscious Statistician (LOTUS)

Random variable X takes values in $\mathcal{X}$. Then for any function g:

$$E(g(X)) = \sum_{x \in \mathcal{X}} p_X(x) g(x),$$

that is, average the function values weighted by the PMF of X.

Law Of The Unconscious Statistician (LOTUS)

Random variable X takes values in $\mathcal{X}$. Then for any function g:

$$E(g(X)) = \sum_{x \in \mathcal{X}} p_X(x) g(x),$$

that is, average the function values weighted by the PMF of X.

Proof.

Group the sum according to the value of X:

$$\begin{aligned} E(g(X)) &= \sum_{\omega \in \Omega} P(\{\omega\})\, g(X(\omega)) \\ &= \sum_{x \in \mathcal{X}} \sum_{\omega \in \Omega : X(\omega) = x} P(\{\omega\})\, g(x) \\ &= \sum_{x \in \mathcal{X}} \underbrace{\left(\sum_{\omega \in \Omega : X(\omega) = x} P(\{\omega\}) \right)}_{P(X=x)} g(x) \end{aligned}$$

□

Summary

Utility of PMF: Find

- probability of events: $P(X \in I)$
- conditional probabilities: $P(X \in I | X \in J)$
- PMF of $g(X)$
- Expectation of $g(X)$
- ...

Independence of random variables

Independence of two random variables

Recall independence of two events:

$$P(A \cap B) = P(A)P(B)$$

Independence of two random variables

Recall independence of two events:

$$P(A \cap B) = P(A)P(B)$$

Definition

Discrete random variables X and Y are independent if **for any** $x, y \in \mathbb{R}$, $\{X = x\}$ and $\{Y = y\}$ are independent events, i.e.,

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

Independence of two random variables

Recall independence of two events:

$$P(A \cap B) = P(A)P(B)$$

Definition

Discrete random variables X and Y are independent if **for any** $x, y \in \mathbb{R}$, $\{X = x\}$ and $\{Y = y\}$ are independent events, i.e.,

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

Equivalently: conditioning does not change PMF

$$P(X = x | Y = y) = P(X = x)$$

provided that $P(Y = y) > 0$.

Consequences

X and Y independent $\implies$ any event involving X is independent of that involving Y: for any $I, J \subset \mathbb{R}$,

$$P(X \in I, Y \in J) = P(X \in I)P(Y \in J)$$

Consequences

X and Y independent $\implies$ any event involving X is independent of that involving Y: for any $I, J \subset \mathbb{R}$,

$$P(X \in I, Y \in J) = P(X \in I)P(Y \in J)$$

Proof.

$$\begin{aligned}
P(X \in I, Y \in J) &= \sum_{x \in I} \sum_{y \in J} P(X = x, Y = y) \\
&= \sum_{x \in I} \sum_{y \in J} P(X = x)P(Y = y) && \text{independence} \\
&= \underbrace{\left(\sum_{x \in I} P(X = x)\right)}_{P(X \in I)} \underbrace{\left(\sum_{y \in J} P(Y = y)\right)}_{P(Y \in J)}
\end{aligned}$$

□

Consequences

If X and Y are independent, then

$$E(XY) = E(X)E(Y)$$

Consequences

If X and Y are independent, then

$$E(XY) = E(X)E(Y)$$

Proof.

$$\begin{aligned}
E(XY) &= \sum_x \sum_y xyP(X = x, Y = y) && \text{LOTUS} \\
&= \sum_x \sum_y xyP(X = x)P(Y = y) && \text{independence} \\
&= \underbrace{\left(\sum_x xP(X = x) \right)}_{E(X)} \underbrace{\left(\sum_y yP(Y = y) \right)}_{E(Y)}
\end{aligned}$$

□

Summary

- For any X and Y,

$$E(X+Y) = E(X) + E(Y)$$

- For independent X and Y,

$$E(XY) = E(X)E(Y)$$

Consequences

If X and Y are independent, then

- $f(X)$ and $g(Y)$ are also independent
- Hence
$$E(f(X)g(Y)) = E(f(X))E(g(Y))$$

Example: fair die

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$

Example: fair die

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$

Example: fair die

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- Are Y and Z independent?

Example: fair die

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- Are Y and Z independent? Yes!

Example: fair die

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- Are Y and Z independent? Yes!

$$P(Y = 0, Z = 0) = P(X = 6) = 1/6$$

Example: fair die

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- Are Y and Z independent? Yes!

$$P(Y = 0, Z = 0) = P(X = 6) = 1/6$$
$$P(Y = 0, Z = 1) = P(X = 4) = 1/6$$
$$\cdots\cdots$$
$$P(Y = 1, Z = 2) = P(X = 5) = 1/6$$

Example

Are X and $g(X)$ independent?

Example

Are X and $g(X)$ independent? In general no, e.g.,

- $X = \pm 1$ equally likely, $Y = -X$
- $P(X = 1, Y = 1) = 0 \neq P(X = 1)P(Y = 1) = \frac{1}{4}$

Example

Are X and $g(X)$ independent? In general no, e.g.,

- $X = \pm 1$ equally likely, $Y = -X$
- $P(X = 1, Y = 1) = 0 \neq P(X = 1)P(Y = 1) = \frac{1}{4}$

unless in trivial cases (e.g. X is a constant or g is a constant function)

Independence of multiple random variables

Definition

Discrete random variables $X_1, \ldots, X_n$ are independent if for any $x_1, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \times \cdots \times P(X_n = x_n).$$

Independence of multiple random variables

Definition

Discrete random variables $X_1, \ldots, X_n$ are independent if for any $x_1, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \times \cdots \times P(X_n = x_n).$$

Consequences:

- X_1 and $X_2 + X_3$ are independent
- X_4^2 and $X_5 X_6 - X_7$ are independent
- ...

Conditional independence

Independence (of events and random variables) naturally extends to conditional scenarios (B-H: Def 2.5.7 and 3.8.10):

Definition

Let C be an event with $P(C) > 0$.

- **Events** A and B are conditionally independent given an **event** C if $P(A \cap B|C) = P(A|C)P(B|C)$

Conditional independence

Independence (of events and random variables) naturally extends to conditional scenarios (B-H: Def 2.5.7 and 3.8.10):

Definition

Let C be an event with $P(C) > 0$.

- **Events** A and B are conditionally independent given an **event** C if $P(A \cap B|C) = P(A|C)P(B|C)$
- **Random variables** X and Y are conditionally independent given an **event** C if

$$P(X = x, Y = y|C) = P(X = x|C)P(Y = y|C)$$

  for all x, y

Conditional independence

Independence (of events and random variables) naturally extends to conditional scenarios (B-H: Def 2.5.7 and 3.8.10):

Definition

Let C be an event with $P(C) > 0$.

- **Events** A and B are conditionally independent given an **event** C if $P(A \cap B|C) = P(A|C)P(B|C)$
- **Random variables** X and Y are conditionally independent given an **event** C if

$$P(X = x, Y = y|C) = P(X = x|C)P(Y = y|C)$$

  for all x, y
- **Random variables** X and Y are conditionally independent given an **random variable** Z if

$$P(X = x, Y = y|Z = z) = P(X = x|Z = z)P(Y = y|Z = z)$$

  for all x, y and for all z such that $P(Z = z) > 0$

Example: fair die (continued)

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- We have verified that Y and Z independent.

Example: fair die (continued)

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- We have verified that Y and Z independent.
- Questions:
  - Are Y and Z independent conditioned on the event $\{X \leq 3\}$?

Example: fair die (continued)

Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- We have verified that Y and Z independent.
- Questions:
  - Are Y and Z independent conditioned on the event $\{X \le 3\}$? No.
    - For example, $P(Y = 0, Z = 1 | X \le 3) = 0$, but $P(Y = 0 | X \le 3) = P(Z = 1 | X \le 3) = \frac{1}{3}$

Example: fair die (continued)

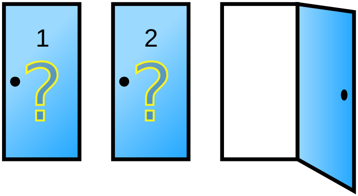
Let X be the result of a fair die. Let

$$Y = X \mod 2$$
$$Z = X \mod 3$$

Then

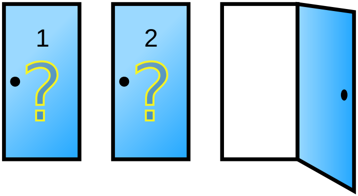
- $p_Y(0) = p_Y(1) = 1/2$
- $p_Z(0) = p_Z(1) = p_Z(2) = 1/3$
- We have verified that Y and Z independent.
- Questions:
  - Are Y and Z independent conditioned on the event $\{X \leq 3\}$? No.
    - For example, $P(Y = 0, Z = 1 | X \leq 3) = 0$, but $P(Y = 0 | X \leq 3) = P(Z = 1 | X \leq 3) = \frac{1}{3}$
  - Are Y and Z independent conditioned on the event $\{X \text{ is even}\}$? Yes. (Exercise)

Monty Hall problem



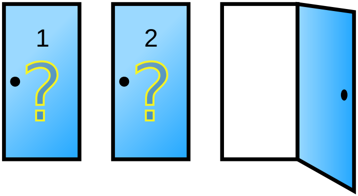
- A prize is behind one of the three doors. The other two are empty

Monty Hall problem



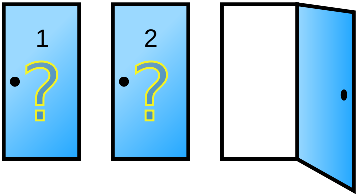
- A prize is behind one of the three doors. The other two are empty
- The guest randomly chooses a door

Monty Hall problem



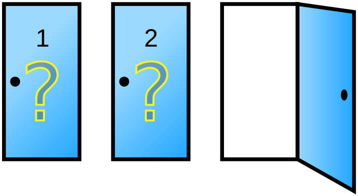
- A prize is behind one of the three doors. The other two are empty
- The guest randomly chooses a door
- The host, knowing where the prize is, opens one of the remaining two doors that is empty.

Monty Hall problem



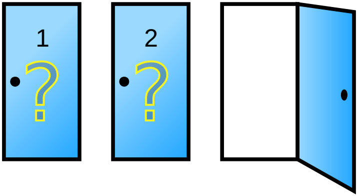
- A prize is behind one of the three doors. The other two are empty
- The guest randomly chooses a door
- The host, knowing where the prize is, opens one of the remaining two doors that is empty. Specifically
  - If guest's door is empty, host opens the other empty door;
  - If guest's door has prize, host randomly chooses an empty door.

Monty Hall problem



- A prize is behind one of the three doors. The other two are empty
- The guest randomly chooses a door
- The host, knowing where the prize is, opens one of the remaining two doors that is empty. Specifically
  - If guest's door is empty, host opens the other empty door;
  - If guest's door has prize, host randomly chooses an empty door.
- The hosts offers the guest the option to switch the choice to the other closed door

Monty Hall problem



- A prize is behind one of the three doors. The other two are empty
- The guest randomly chooses a door
- The host, knowing where the prize is, opens one of the remaining two doors that is empty. Specifically
  - If guest's door is empty, host opens the other empty door;
  - If guest's door has prize, host randomly chooses an empty door.
- The hosts offers the guest the option to switch the choice to the other closed door
- Question: Would you switch? Not switch? Doesn't matter?

Monty Hall problem

Define the relevant random variables

- X: location of prize

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

- X and G are independent and equally likely to be $1, 2, 3$

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

- X and G are independent and equally likely to be $1, 2, 3$
- Then $P(X = G) = \frac{1}{3}$
  (analogy: for two dice $P(\text{double}) = 1/6$ in lecture 2)

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

- X and G are independent and equally likely to be $1, 2, 3$
- Then $P(X = G) = \frac{1}{3}$
  (analogy: for two dice $P(\text{double}) = 1/6$ in lecture 2)
- Per rule of the game, either $X = G$ or $X = S$

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

- X and G are independent and equally likely to be $1, 2, 3$
- Then $P(X = G) = \frac{1}{3}$
  (analogy: for two dice $P(\text{double}) = 1/6$ in lecture 2)
- Per rule of the game, either $X = G$ or $X = S$
- Hence $P(X = S) = \frac{2}{3}$, i.e., switching is twice as likely to win

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

- X and G are independent and equally likely to be $1, 2, 3$
- Then $P(X = G) = \frac{1}{3}$
  (analogy: for two dice $P(\text{double}) = 1/6$ in lecture 2)
- Per rule of the game, either $X = G$ or $X = S$
- Hence $P(X = S) = \frac{2}{3}$, i.e., switching is twice as likely to win
- Another way for guest to think:

Monty Hall problem

Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

- X and G are independent and equally likely to be $1, 2, 3$
- Then $P(X = G) = \frac{1}{3}$
  (analogy: for two dice $P(\text{double}) = 1/6$ in lecture 2)
- Per rule of the game, either $X = G$ or $X = S$
- Hence $P(X = S) = \frac{2}{3}$, i.e., switching is twice as likely to win
- Another way for guest to think:
  - If my door is empty ($2/3$ chance), switch will win

Monty Hall problem

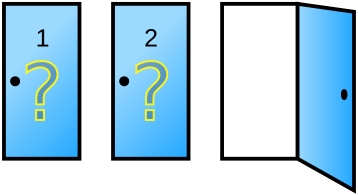
Define the relevant random variables

- X: location of prize
- G: choice of guest
- H: choice of host
- S: the other door

Then

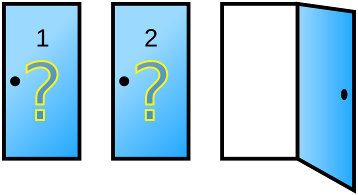
- X and G are independent and equally likely to be $1, 2, 3$
- Then $P(X = G) = \frac{1}{3}$
  (analogy: for two dice $P(\text{double}) = 1/6$ in lecture 2)
- Per rule of the game, either $X = G$ or $X = S$
- Hence $P(X = S) = \frac{2}{3}$, i.e., switching is twice as likely to win
- Another way for guest to think:
  - If my door is empty ($2/3$ chance), switch will win
  - If my door has prize ($1/3$ chance), switch will lose

Monty Hall problem



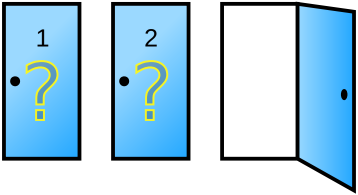
- Maybe you are still not convinced because the guest knows which door he chooses and observes which door the hosts opens

Monty Hall problem



- Maybe you are still not convinced because the guest knows which door he chooses and observes which door the hosts opens
- The guest should evaluate the probability conditioned on these information in order to make an informed decision.

Monty Hall problem



- Maybe you are still not convinced because the guest knows which door he chooses and observes which door the hosts opens
- The guest should evaluate the probability conditioned on these information in order to make an informed decision.
- So let's compute some conditional probabilities

Monty Hall problem: conditional probability

Suppose guest chooses first door and host opens third door, i.e., $G = 1$ and $H = 3$. Goal: find

$$P(X = 1 | G = 1, H = 3) \quad \text{and} \quad P(X = 2 | G = 1, H = 3)$$

Monty Hall problem: conditional probability

Suppose guest chooses first door and host opens third door, i.e., $G = 1$ and $H = 3$. Goal: find

$$P(X = 1|G = 1, H = 3) \quad \text{and} \quad P(X = 2|G = 1, H = 3)$$

Use law of total probability and Bayes rule:

$$\begin{aligned} P(G = 1, H = 3) = &\underbrace{P(G = 1, H = 3|X = 1)}_{1/3\times 1/2}\underbrace{P(X = 1)}_{1/3} \\ &+ \underbrace{P(G = 1, H = 3|X = 2)}_{1/3\times 1}\underbrace{P(X = 2)}_{1/3} \\ &+ \underbrace{P(G = 1, H = 3|X = 3)}_{1/3\times 0}\underbrace{P(X = 3)}_{1/3} \\ = &\frac{1}{6} \end{aligned}$$

and hence

Monty Hall problem: conditional probability

Suppose guest chooses first door and host opens third door, i.e., $G = 1$ and $H = 3$. Goal: find

$$P(X = 1|G = 1, H = 3) \quad \text{and} \quad P(X = 2|G = 1, H = 3)$$

Use law of total probability and Bayes rule:

$$\begin{aligned} P(G = 1, H = 3) = & \underbrace{P(G = 1, H = 3|X = 1)}_{1/3\times 1/2}\underbrace{P(X = 1)}_{1/3} \\ & + \underbrace{P(G = 1, H = 3|X = 2)}_{1/3\times 1}\underbrace{P(X = 2)}_{1/3} \\ & + \underbrace{P(G = 1, H = 3|X = 3)}_{1/3\times 0}\underbrace{P(X = 3)}_{1/3} \\ = & \frac{1}{6} \end{aligned}$$

and hence

$$P(X = 1|G = 1, H = 3) = \frac{1}{3} \quad \text{and} \quad P(X = 2|G = 1, H = 3) = \frac{2}{3}$$