S&DS 241 Lecture 7

Union bound. Inclusion-Exclusion principles.

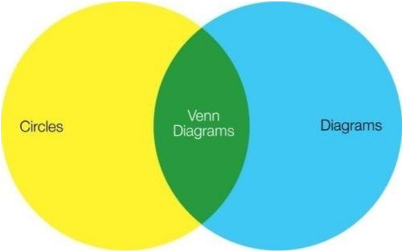
B-H: 1.6,4.4

Recall: Union of two events

From axioms of probability:

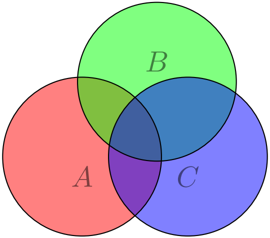
$$P(A \cup B) = P(A) + P(B) - P(AB)$$

(Again: we omit $\cap$ and write AB for intersection)



Recall: Union of three events

$$
\begin{aligned}
P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\
& - P(AB) - P(BC) - P(CA) \\
& + P(ABC)
\end{aligned}
$$



Today

How to deal with

$$P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right)$$

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We will learn:

1. Inequality: union bound (Boole's or Bonferroni's inequality)
2. Equality: inclusion-exclusion principle

Union bound (B-H: Example 4.4.3)

- Two events:

$$P(A \cup B) \leq P(A) + P(B)$$

Proof: $P(A \cup B) = P(A) + P(B) - P(AB)$

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- Corollary:

$$P(A_1 \cup \ldots \cup A_n) \leq P(A_1) + \ldots + P(A_n)$$

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- Convenient to use: no need to deal with intersections or postulate independence

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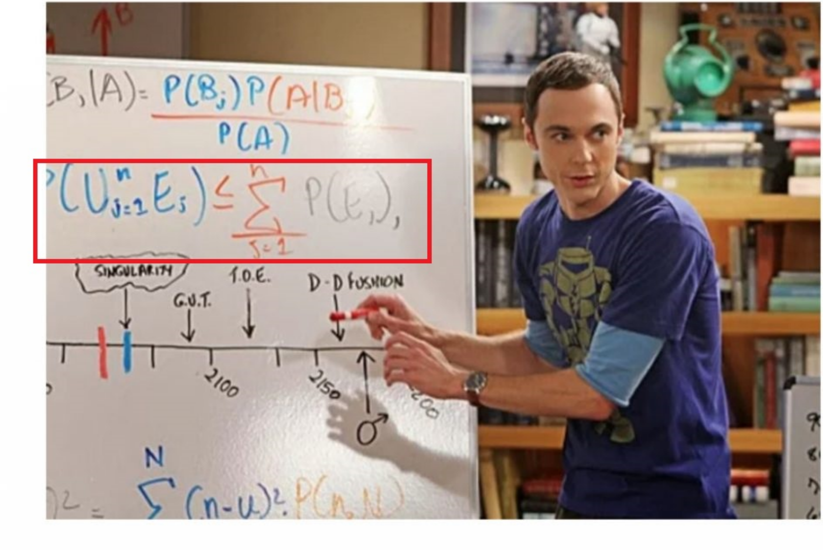
- Corollary:

$$P(A_1 \cup \ldots \cup A_n) \leq P(A_1) + \ldots + P(A_n)$$

Remarks

- Convenient to use: no need to deal with intersections or postulate independence
- Conservative estimate and can be useless (might exceed 1)


SINGULARITY
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Example

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$$P(\text{fails at least one class})$$

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$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - P(A_1^c A_2^c A_3^c A_4^c) = 1 - (1 - 3\%)^4 \approx 11.5\%$$

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- Independence might not be realistic to assume. But union bound always applies.

Inclusion-Exclusion Principle

$$
\begin{aligned}
& P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) \\
= & \sum_{i=1}^n P\left(A_i\right) \\
& -\sum_{1 \leq i<j \leq n}^n P\left(A_i A_j\right) \\
& +\sum_{1 \leq i<j<k \leq n}^n P\left(A_i A_j A_k\right) \\
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& +(-1)^{n-1} P\left(A_1 A_2 \cdots A_n\right) .
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Proof 1 induction on n (exercise)

Inclusion-Exclusion Principle

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Proof 1 induction on n (exercise)

Proof 2 indicator random variables (later)

Example: Matching

A hat-checker in a restaurant, having checked n hats, gets them hopelessly scrambled and returns them at random to the guests as they leave. Let $X =$ number of guests with own hats. Find $E(X)$ without finding PMF.

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$$A_i = \{i^{\text{th}} \text{ guest gets own hat}\}, \quad i = 1, \ldots, n$$

Then $P(A_i) = 1/n$

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$$X_i = \mathbf{1}_{A_i} = \begin{cases} 1 & i^{\text{th}} \text{ guest gets own hat} \\ 0 & \text{otherwise} \end{cases}$$

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- Note $X = X_1 + \cdots + X_n$ and hence

$$E(X) = E(X_1) + \cdots + E(X_n) = P(A_1) + \cdots + P(A_n) = \frac{1}{n} + \cdots + \frac{1}{n} = 1$$

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- Recall

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- Note

$$\{X = 0\} = \{\text{nobody gets own hat}\} = A_1^c \cap A_2^c \cdots \cap A_n^c$$
$$\{X > 0\} = \{\text{somebody gets own hat}\} = A_1 \cup A_2 \cdots \cup A_n$$

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- Plan: use inclusion-exclusion principle to find $P(A_1 \cup A_2 \cdots \cup A_n)$.
- Union bound is not useful here:

$$P(A_1 \cup A_2 \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots P(A_n) = 1$$

Inclusion-exclusion principle

$$
\begin{aligned}
& P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) \\
=& \sum_{i=1}^{n} P\left(A_i\right) && n \text{ terms} \\
& - \sum_{1 \le i < j \le n}^{n} P\left(A_i A_j\right) && \text{all } \binom{n}{2} \text{ pairs} \\
& + \sum_{1 \le i < j < k \le n}^{n} P\left(A_i A_j A_k\right) && \text{all } \binom{n}{3} \text{ triples} \\
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\end{aligned}
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Inclusion-exclusion principle

$$P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right)$$

$$= \sum_{i=1}^{n} P\left(A_i\right) \qquad n \text{ terms}$$

$$- \sum_{1 \le i < j \le n}^{n} P\left(A_i A_j\right) \qquad \text{all } \binom{n}{2} \text{ pairs}$$

$$+ \sum_{1 \le i < j < k \le n}^{n} P\left(A_i A_j A_k\right) \qquad \text{all } \binom{n}{3} \text{ triples}$$

$$- \cdots$$

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We already know: $P(A_i) = 1/n$,

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We already know: $P(A_i) = 1/n$, or another way: $P(A_i) = \frac{(n-1)!}{n!}$

Pairwise intersection

$$P(A_iA_j) = P(A_i)P(A_j|A_i) = \frac{1}{n} \times \frac{1}{n-1}$$

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Total number of pairs: $\binom{n}{2} = \frac{n!}{2!(n-2)!}$

Triple intersection

$$P(A_iA_jA_k) = P(A_i)P(A_j|A_i)P(A_k|A_iA_j) = \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$$

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Total number of pairs: $\binom{n}{3} = \frac{n!}{3!(n-3)!}$

Intersection of all

$$P\left(A_1 A_2 \cdots A_n\right) = \frac{1}{n!}$$

Inclusion-exclusion principle

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Inclusion-exclusion principle

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\begin{aligned}
& P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) \\
=& 1 \\
& - \binom{n}{2} \times \frac{(n-2)!}{n!} \\
& + \binom{n}{3} \times \frac{(n-3)!}{n!} \\
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& -\cdots \\
& +(-1)^{n-1} \frac{1}{n!} \\
= & 1-\frac{1}{2!}+\frac{1}{3!}-\cdots+(-1)^{n-1} \frac{1}{n!}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& P(X=0) \\
= & 1 - P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) \\
= & \frac{1}{2!} - \frac{1}{3!} + \cdots - (-1)^{n-1}\frac{1}{n!}
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If there are many people, $P(\text{nobody gets own hat}) \approx 1/e$

Why?

- Taylor expansion: [B-H: Math Appendix A.8.3]

$$
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
$$

- Expand at $x=-1$: $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$

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- Taylor expansion: [B-H: Math Appendix A.8.3]

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- Expand at $x = -1$: $e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$

What about $P(X = 1)$?

Is $1/e$ surprising?

- Let's suppose, hypothetically, A_i's were independent. Then the probability of nobody gets own hat would be

$$\left(1 - \frac{1}{n}\right)^n \xrightarrow{n\to\infty} \frac{1}{e}$$

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- Of course they are dependent, for example,
  - the 1st person getting own hat increases the chance that 2nd person gets own hat (Why?)
  - $P(A_2|A_1) = \frac{1}{n-1} > P(A_2) = \frac{1}{n}$
- Nevertheless, the dependence is rather weak when n is large, and inclusion-exclusion principle makes it possible to rigorously compute it.

Let's prove the inclusion-exclusion principle

Recall: indicator random variable

- For every event A, we define a binary-valued random variable:

$$\mathbf{1}_A = \begin{cases} 1 & A \text{ occurs} \\ 0 & A \text{ does not occur} \end{cases}$$

- Important relation ("fundamental bridge"):

$$E(\mathbf{1}_A) = P(A)$$

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- Calculus of indicators (shorthand: $AB = A \cap B$)
  - $\mathbf{1}_A \times \mathbf{1}_B = \mathbf{1}_{AB}$
  - $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$

Proof of the Inclusion and Exclusion Principle

$$\mathbf{1}_{A_1 \cup A_2 \cup \cdots \cup A_n}$$

Proof of the Inclusion and Exclusion Principle

$$\mathbf{1}_{A_1 \cup A_2 \cup \cdots \cup A_n} = 1 - \mathbf{1}_{A_1^c A_2^c \cdots A_n^c}$$

Proof of the Inclusion and Exclusion Principle

$$\begin{aligned}\mathbf{1}_{A_1\cup A_2\cup\cdots\cup A_n} =&1-\mathbf{1}_{A_1^cA_2^c\cdots A_n^c}\\ =&1-\mathbf{1}_{A_1^c}\times\mathbf{1}_{A_2^c}\times\cdots\times\mathbf{1}_{A_n^c}\end{aligned}$$

Proof of the Inclusion and Exclusion Principle

$$\begin{aligned}
\mathbf{1}_{A_1 \cup A_2 \cup \cdots \cup A_n} =& 1 - \mathbf{1}_{A_1^c A_2^c \cdots A_n^c} \\
=& 1 - \mathbf{1}_{A_1^c} \times \mathbf{1}_{A_2^c} \times \cdots \times \mathbf{1}_{A_n^c} \\
=& 1 - \left(1 - \mathbf{1}_{A_1}\right)\left(1 - \mathbf{1}_{A_2}\right) \cdot \cdots \cdot \left(1 - \mathbf{1}_{A_n}\right)
\end{aligned}$$

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$$
\begin{aligned}
\mathbf{1}_{A_1 \cup A_2 \cup \cdots \cup A_n} =& 1 - \mathbf{1}_{A_1^c A_2^c \cdots A_n^c} \\
=& 1 - \mathbf{1}_{A_1^c} \times \mathbf{1}_{A_2^c} \times \cdots \times \mathbf{1}_{A_n^c} \\
=& 1 - \left(1 - \mathbf{1}_{A_1}\right)\left(1 - \mathbf{1}_{A_2}\right) \cdot \cdots \cdot \left(1 - \mathbf{1}_{A_n}\right) \\
=& \sum_{i=1}^{n} \mathbf{1}_{A_i} \\
& - \sum_{1 \le i < j \le n}^{n} \mathbf{1}_{A_i A_j} \\
& + \sum_{1 \le i < j < k \le n}^{n} \mathbf{1}_{A_i A_j A_k} \\
& - \cdots \\
& + (-1)^{n-1} \mathbf{1}_{A_1 A_2 \cdots A_n}
\end{aligned}
$$

Proof of the Inclusion and Exclusion Principle

Take expectation on both sides and invoke linearity of expectation:

$$\begin{aligned} E(\mathbf{1}_{A_1\cup A_2\cup\cdots\cup A_n}) &= \sum_{i=1}^{n} E(\mathbf{1}_{A_i}) \\ &\quad - \sum_{1\le i<j\le n}^{n} E(\mathbf{1}_{A_iA_j}) \\ &\quad + \sum_{1\le i<j<k\le n}^{n} E(\mathbf{1}_{A_iA_jA_k}) \\ &\quad - \cdots \\ &\quad + (-1)^{n-1} E(\mathbf{1}_{A_1A_2\cdots A_n}) \end{aligned}$$

Proof of the Inclusion and Exclusion Principle

Thus

$$
\begin{aligned}
P\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) = & \sum_{i=1}^{n} P\left(A_i\right) \\
& - \sum_{1 \leq i < j \leq n}^{n} P\left(A_i A_j\right) \\
& + \sum_{1 \leq i < j < k \leq n}^{n} P\left(A_i A_j A_k\right) \\
& - \cdots \\
& + (-1)^{n-1} P\left(A_1 A_2 \cdots A_n\right).
\end{aligned}
$$

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- Using similar reasoning, we can show (Exercise)

$$P\left(A_1 \cup \ldots \cup A_n\right) \geq P\left(A_1\right) + \ldots + P\left(A_n\right) - \sum_{i<j} P(A_i A_j)$$