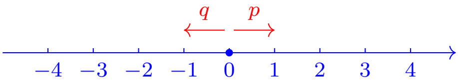
S&DS 241 Lecture 10

Random walk: Gambler's ruin

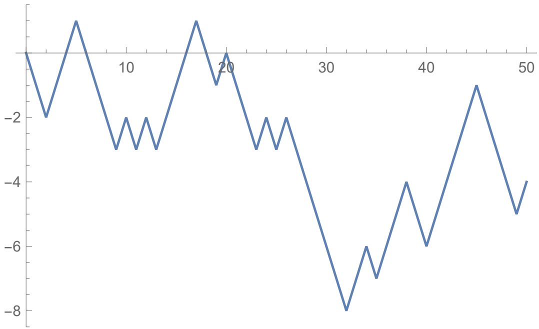
B-H: 2.7 (Example 2.7.3), math appendix A.4,A.8

Random walk

A particle starts at 0, and at each step it either moves 1 unit to the right with probability p or to the left with probability $q = 1 - p$, independently.



Let S_n be the particle's position after n steps.



PMF of S_n

Let

$$Z_i = i^{\text{th}} \text{ step} = \begin{cases} +1 & \text{with prob } p \\ -1 & \text{with prob } q \end{cases}$$

Then

$$S_n = \underbrace{Z_1 + \cdots + Z_n}_{\text{iid}}$$

PMF of S_n

Let

$$Z_i = i^{\text{th}} \text{ step} = \begin{cases} +1 & \text{with prob } p \\ -1 & \text{with prob } q \end{cases}$$

Then

$$S_n = \underbrace{Z_1 + \cdots + Z_n}_{\text{iid}}$$

To relate to binomial,

- Let $X=$ number of steps to the right $\sim \text{Bin}(n, p)$

PMF of S_n

Let

$$Z_i = i^{\text{th}} \text{ step} = \begin{cases} +1 & \text{with prob } p \\ -1 & \text{with prob } q \end{cases}$$

Then

$$S_n = \underbrace{Z_1 + \cdots + Z_n}_{\text{iid}}$$

To relate to binomial,

- Let X= number of steps to the right $\sim \text{Bin}(n, p)$
- Then $S_n = 2X - n \in \{-n, -n+2, \ldots, n-2, n\}$ and

$$P(S_n = j) = P(X = (n+j)/2) = \binom{n}{\frac{n+j}{2}} p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}$$

PMF of S_n

Let

$$Z_i = i^{\text{th}} \text{ step} = \begin{cases} +1 & \text{with prob } p \\ -1 & \text{with prob } q \end{cases}$$

Then

$$S_n = \underbrace{Z_1 + \cdots + Z_n}_{\text{iid}}$$

To relate to binomial,

- Let $X =$ number of steps to the right $\sim \text{Bin}(n, p)$
- Then $S_n = 2X - n \in \{-n, -n+2, \ldots, n-2, n\}$ and

$$P(S_n = j) = P(X = (n+j)/2) = \binom{n}{\frac{n+j}{2}} p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}$$

- Special case: symmetric random walk ($p = 1/2$)

$$P(S_n = j) = P(X = (n+j)/2) = \binom{n}{\frac{n+j}{2}} 2^{-n}$$

Gambler's ruin

Two gamblers, with a bankroll of $\$3$ and $\$7$, respectively, bet on independent tosses of a fair coin. The first gambler wins $\$1$ if a toss is head; the second gambler wins $\$1$ if a toss is tail. The game stops if either runs out of money. What is the probability that the first gambler wins all the money?

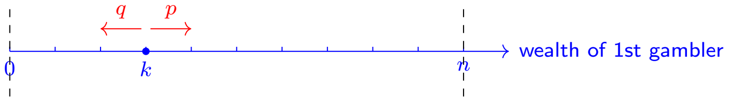
Gambler's ruin: general version

Two gamblers, with a bankroll of $\$k$ and $\$n - k$, respectively, bet on independent tosses of a coin. The first gambler wins $\$1$ if a toss is head, with probability p; the second gambler wins $\$1$ if a toss is tail with probability $q = 1 - p$. The game stops if either runs out of money. What is the probability that the first gambler wins all the money?

Gambler's ruin: general version

Two gamblers, with a bankroll of $\$k$ and $\$n-k$, respectively, bet on independent tosses of a coin. The first gambler wins $\$1$ if a toss is head, with probability p; the second gambler wins $\$1$ if a toss is tail with probability $q = 1 - p$. The game stops if either runs out of money. What is the probability that the first gambler wins all the money?

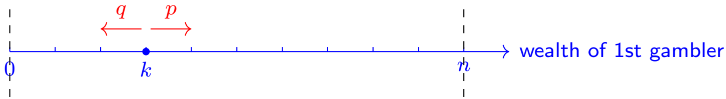
- In the language of random walk: start the walk at k, what is the probability that the particle hits 0 before hitting n?



Gambler's ruin: general version

Two gamblers, with a bankroll of $\$k$ and $\$n-k$, respectively, bet on independent tosses of a coin. The first gambler wins $\$1$ if a toss is head, with probability p; the second gambler wins $\$1$ if a toss is tail with probability $q = 1 - p$. The game stops if either runs out of money. What is the probability that the first gambler wins all the money?

- In the language of random walk: start the walk at k, what is the probability that the particle hits 0 before hitting n?



Goal

- How does winning probability depends on the initial wealth?
- How does winning probability depends on the chance of each toss?

Special case: Tennis ($k = n - k = 2$)

(Lecture 4) Alice plays tennis against Bob. The game is at deuce. Suppose

- Alice wins each point with probability p and loses with probability $q = 1 - p$
- Each point is played independently
- The game is won by the player who leads by 2 points

Special case: Tennis ($k = n - k = 2$)

(Lecture 4) Alice plays tennis against Bob. The game is at deuce.
Suppose

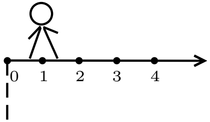
- Alice wins each point with probability p and loses with probability $q = 1 - p$
- Each point is played independently
- The game is won by the player who leads by 2 points

Solution:

$$\boxed{P(\text{Alice eventually wins}) = \frac{p^2}{p^2 + q^2}}$$

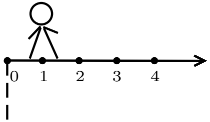
Special case: Cliff ($k = 1, n = \infty$)

(PSet 3) A drunkard is standing one step away from the cliff on his left. He moves randomly, one step at a time and independently, either to the right (away from the cliff) with probability p or left (toward the cliff) with probability $1 - p$.



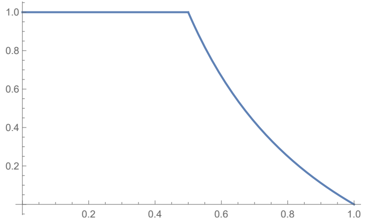
Special case: Cliff ($k = 1, n = \infty$)

(PSet 3) A drunkard is standing one step away from the cliff on his left. He moves randomly, one step at a time and independently, either to the right (away from the cliff) with probability p or left (toward the cliff) with probability $1 - p$.



Solution:

$$P(\text{eventually falls down the cliff}) = \begin{cases} 1 & 0 \leq p \leq \frac{1}{2} \\ \frac{1-p}{p} & \frac{1}{2} \leq p \leq 1 \end{cases}$$



Fair coin: $p = 1/2$

Winning probabilities

Let P_k be the probability that the first gambler eventually wins if he starts with a bankroll of $\$k$ and his opponent starts with $\$n-k$, i.e.,

$$P_k = P(\text{1st gambler eventually wins all the money starting with } \$k)$$
$$Q_k = P(\text{1st gambler eventually loses all the money starting with } \$k)$$

Winning probabilities

Let P_k be the probability that the first gambler eventually wins if he starts with a bankroll of $\$k$ and his opponent starts with $\$n-k$, i.e.,

$$P_k = P(\text{1st gambler eventually wins all the money starting with } \$k)$$
$$Q_k = P(\text{1st gambler eventually loses all the money starting with } \$k)$$

Winning probabilities

Let P_k be the probability that the first gambler eventually wins if he starts with a bankroll of $\$k$ and his opponent starts with $\$n-k$, i.e.,

$$P_k = P(\text{1st gambler eventually wins all the money starting with } \$k)$$
$$Q_k = P(\text{1st gambler eventually loses all the money starting with } \$k)$$

- Question: Is it true that $P_k + Q_k = 1$?

Winning probabilities

Let P_k be the probability that the first gambler eventually wins if he starts with a bankroll of $\$k$ and his opponent starts with $\$n-k$, i.e.,

$$P_k = P(\text{1st gambler eventually wins all the money starting with } \$k)$$
$$Q_k = P(\text{1st gambler eventually loses all the money starting with } \$k)$$

- Question: Is it true that $P_k + Q_k = 1$?
- It is unclear from first principles that $P_k + Q_k = 1$, because the third possibility is that the game never ends (keep going back and forth)!

Winning probabilities

Let P_k be the probability that the first gambler eventually wins if he starts with a bankroll of $\$k$ and his opponent starts with $\$n-k$, i.e.,

$$P_k = P(\text{1st gambler eventually wins all the money starting with } \$k)$$
$$Q_k = P(\text{1st gambler eventually loses all the money starting with } \$k)$$

- Question: Is it true that $P_k + Q_k = 1$?
- It is unclear from first principles that $P_k + Q_k = 1$, because the third possibility is that the game never ends (keep going back and forth)!
- It turns out that the game ends with probability one.

Winning probabilities

Let P_k be the probability that the first gambler eventually wins if he starts with a bankroll of $\$k$ and his opponent starts with $\$n-k$, i.e.,

$$P_k = P(\text{1st gambler eventually wins all the money starting with } \$k)$$
$$Q_k = P(\text{1st gambler eventually loses all the money starting with } \$k)$$

- Question: Is it true that $P_k + Q_k = 1$?
- It is unclear from first principles that $P_k + Q_k = 1$, because the third possibility is that the game never ends (keep going back and forth)!
- It turns out that the game ends with probability one.
- Let's first compute P_k. By definition: $P_n = 1$ and $P_0 = 0$.
- Next: find a recursion for P_k

Law of total probability

Gambling process

- If the next toss is head, 1st gambler then has $\$(k+1)$
- If the next toss is tail, 1st gambler then has $\$(k-1)$

Law of total probability

Gambling process

- If the next toss is head, 1st gambler then has $\$(k+1)$
- If the next toss is tail, 1st gambler then has $\$(k-1)$

Therefore

$$P_k = \frac{1}{2}\underbrace{P(\text{1st gambler wins starting with } \$k+1 | \text{next toss head})}_{P_{k+1}}$$
$$+ \frac{1}{2}\underbrace{P(\text{1st gambler wins starting with } \$k-1 | \text{next toss tail})}_{P_{k-1}}$$

Difference equation (B-H Math Appendix A.5)

$$\begin{cases} P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1}, \quad k = 1, \ldots, n-1 & \text{(recursion)} \\ P_0 = 0, P_n = 1 & \text{(boundary conditions)} \end{cases}$$

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then

$$P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1}$$

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then

$$P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1}$$
$$\iff \frac{1}{2}P_k + \frac{1}{2}P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1}$$

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then

$$\begin{aligned} P_k &= \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1} \\ \iff \frac{1}{2}P_k + \frac{1}{2}P_k &= \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1} \\ \iff \underbrace{P_k - P_{k-1}}_{\Delta_{k-1}} &= \underbrace{P_{k+1} - P_k}_{\Delta_k} \end{aligned}$$

Solving difference equation: focus on the increment

For $1 \le k \le n-1$, then

$$
\begin{aligned}
& P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1} \\
\iff & \frac{1}{2}P_k + \frac{1}{2}P_k = \frac{1}{2}P_{k-1} + \frac{1}{2}P_{k+1} \\
\iff & \underbrace{P_k - P_{k-1}}_{\Delta_{k-1}} = \underbrace{P_{k+1} - P_k}_{\Delta_k} \\
\iff & \Delta_0 = \Delta_1 = \cdots = \Delta_{n-1}
\end{aligned}
$$

Solving difference equation: focus on the increment

Recall boundary conditions: $P_0 = 0, P_n = 1$. Then

$$1 = P_n = \underbrace{P_n - P_{n-1}}_{\Delta_{n-1}} + \underbrace{P_{n-1} - P_{n-2}}_{\Delta_{n-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0} = n\Delta_0$$

Solving difference equation: focus on the increment

Recall boundary conditions: $P_0 = 0, P_n = 1$. Then

$$1 = P_n = \underbrace{P_n - P_{n-1}}_{\Delta_{n-1}} + \underbrace{P_{n-1} - P_{n-2}}_{\Delta_{n-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0} = n\Delta_0$$

and

$$P_k = \underbrace{P_k - P_{k-1}}_{\Delta_{k-1}} + \underbrace{P_{k-1} - P_{k-2}}_{\Delta_{k-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0} = k\Delta_0$$

Solving difference equation: focus on the increment

Recall boundary conditions: $P_0 = 0, P_n = 1$. Then

$$1 = P_n = \underbrace{P_n - P_{n-1}}_{\Delta_{n-1}} + \underbrace{P_{n-1} - P_{n-2}}_{\Delta_{n-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0} = n\Delta_0$$

and

$$P_k = \underbrace{P_k - P_{k-1}}_{\Delta_{k-1}} + \underbrace{P_{k-1} - P_{k-2}}_{\Delta_{k-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0} = k\Delta_0$$

Thus

$$\boxed{P_k = \frac{k}{n}, \quad k = 1, \ldots, n}$$

Winning and losing probability

By symmetry:

$$
\begin{aligned}
Q_k &= P(\text{1st gambler loses all the money starting with } \$k) \\
&= P(\text{2nd gambler wins all the money starting with } \$n-k) \\
&= P_{n-k} = \frac{n-k}{n}
\end{aligned}
$$

Winning and losing probability

By symmetry:

$$\begin{aligned} Q_k &= P(\text{1st gambler loses all the money starting with } \$k) \\ &= P(\text{2nd gambler wins all the money starting with } \$n-k) \\ &= P_{n-k} = \frac{n-k}{n} \end{aligned}$$

So

$$\boxed{P_k = \frac{k}{n}, \quad Q_k = \frac{n-k}{n}, \qquad k = 1, \ldots, n}$$

and

$$P(\text{game never ends}) = 1 - P_k - Q_k = 0$$

Back to example

Two gamblers, with a bankroll of $\$3$ and $\$7$, bet on independent tosses of a fair coin. The first gambler wins $\$1$ if a toss is head; the second wins $\$1$ if a toss is a tail. The game stops if either runs out of money.

$$P(\text{1st gambler wins all the money}) = \frac{3}{10}$$
$$P(\text{2nd gambler wins all the money}) = \frac{7}{10}$$

Playing against casino

A gambler with a bankroll of $\$k$ bet on independent tosses of a fair coin against the casino with $\$\infty$ bankroll. The gambler wins $\$1$ if a toss is head; the casino wins $\$1$ if a toss is a tail. The game stops if either runs out of money.

$$\boxed{\begin{aligned} P(\text{gambler wins}) &= 0 \\ P(\text{casino wins}) &= 1 \end{aligned}}$$

Playing against casino

A gambler with a bankroll of $\$k$ bet on independent tosses of a fair coin against the casino with $\$\infty$ bankroll. The gambler wins $\$1$ if a toss is head; the casino wins $\$1$ if a toss is a tail. The game stops if either runs out of money.

$$\boxed{\begin{aligned} P(\text{gambler wins}) &= 0 \\ P(\text{casino wins}) &= 1 \end{aligned}}$$

- As long as the initial wealth k is finite, the gambler is doomed if the game is fair
- Also explains the cliff problem for $p = 1/2$: $P(\text{eventual fall}) = 1$ regarding of the starting position.

Lesson

> "Millionaires should always gamble, poor men never."
>
> — J. M. Keynes

Lesson

> "Millionaires should always gamble, poor men never."
>
> — J. M. Keynes

> "Gambling is risk-taking. It might be said the owner of a casino gambles, takes risks, but he has the odds in his favour, so that's intelligent gambling. If I wanted to gamble, I'd buy the casino."
>
> — J. P. Getty

Biased coin: $p \neq 1/2$

Law of total probability

Gambling process

- If the next toss is head, 1st gambler then has $\$(k+1)$
- If the next toss is tail, 1st gambler then has $\$(k-1)$

Law of total probability

Gambling process

- If the next toss is head, 1st gambler then has $\$(k+1)$
- If the next toss is tail, 1st gambler then has $\$(k-1)$

Therefore

$$P_k = p\,\underbrace{P(\text{1st gambler wins starting with } \$k+1 | \text{next toss head})}_{P_{k+1}}$$
$$+ q\,\underbrace{P(\text{1st gambler wins starting with } \$k-1 | \text{next toss tail})}_{P_{k-1}}$$

Difference equation

$$\begin{cases} P_k = qP_{k-1} + pP_{k+1}, \quad k = 1, \dots, n-1 & \text{(recursion)} \\ P_0 = 0, P_n = 1 & \text{(boundary conditions)} \end{cases}$$

Next we solve this difference equation (B-H Math Appendix A.4)

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then (recall $q = 1 - p$)

$$P_k = qP_{k-1} + pP_{k+1}$$

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then (recall $q = 1 - p$)

$$P_k = qP_{k-1} + pP_{k+1}$$
$$\iff qP_k + pP_k = qP_{k-1} + pP_{k+1}$$

Solving difference equation: focus on the increment

For $1 \le k \le n-1$, then (recall $q = 1 - p$)

$$\begin{aligned} & P_k = qP_{k-1} + pP_{k+1} \\ \iff & qP_k + pP_k = qP_{k-1} + pP_{k+1} \\ \iff & q\underbrace{(P_k - P_{k-1})}_{\Delta_{k-1}} = p\underbrace{(P_{k+1} - P_k)}_{\Delta_k} \end{aligned}$$

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then (recall $q = 1-p$)

$$\begin{aligned} & P_k = qP_{k-1} + pP_{k+1} \\ \iff & qP_k + pP_k = qP_{k-1} + pP_{k+1} \\ \iff & q\underbrace{(P_k - P_{k-1})}_{\Delta_{k-1}} = p\underbrace{(P_{k+1} - P_k)}_{\Delta_k} \\ \iff & \Delta_k = \frac{q}{p}\Delta_{k-1} \end{aligned}$$

Solving difference equation: focus on the increment

For $1 \leq k \leq n-1$, then (recall $q = 1-p$)

$$\begin{aligned} & P_k = qP_{k-1} + pP_{k+1} \\ \iff & qP_k + pP_k = qP_{k-1} + pP_{k+1} \\ \iff & q\underbrace{(P_k - P_{k-1})}_{\Delta_{k-1}} = p\underbrace{(P_{k+1} - P_k)}_{\Delta_k} \\ \iff & \Delta_k = \frac{q}{p}\Delta_{k-1} \end{aligned}$$

In other words,

$$\Delta_k = \left(\frac{q}{p}\right)^k \Delta_0$$

Solving difference equation: focus on the increment

Recall boundary conditions: $P_0 = 0, P_n = 1$. Then[1]

$$1 = P_n = \underbrace{P_n - P_{n-1}}_{\Delta_{n-1}} + \underbrace{P_{n-1} - P_{n-2}}_{\Delta_{n-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0}$$

$$= \Delta_0 \left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{n-1} \right) = \Delta_0 \frac{1 - \left(\frac{q}{p}\right)^n}{1 - \frac{q}{p}}$$

[1]Recall finite geometric sum: $1 + a + \cdots + a^{n-1} = \frac{1-a^n}{1-a}$ (B-H A.8).

Solving difference equation: focus on the increment

Recall boundary conditions: $P_0 = 0, P_n = 1$. Then[1]

$$1 = P_n = \underbrace{P_n - P_{n-1}}_{\Delta_{n-1}} + \underbrace{P_{n-1} - P_{n-2}}_{\Delta_{n-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0}$$

$$= \Delta_0 \left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{n-1}\right) = \Delta_0 \frac{1 - \left(\frac{q}{p}\right)^n}{1 - \frac{q}{p}}$$

Similarly:

$$P_k = \Delta_0 \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \frac{q}{p}}$$

[1]Recall finite geometric sum: $1 + a + \cdots + a^{n-1} = \frac{1-a^n}{1-a}$ (B-H A.8).

Solving difference equation: focus on the increment

Recall boundary conditions: $P_0 = 0, P_n = 1$. Then[1]

$$1 = P_n = \underbrace{P_n - P_{n-1}}_{\Delta_{n-1}} + \underbrace{P_{n-1} - P_{n-2}}_{\Delta_{n-2}} + \cdots + \underbrace{P_1 - P_0}_{\Delta_0}$$

$$= \Delta_0 \left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{n-1} \right) = \Delta_0 \frac{1 - \left(\frac{q}{p}\right)^n}{1 - \frac{q}{p}}$$

Similarly:

$$P_k = \Delta_0 \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \frac{q}{p}}$$

Thus

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}, \quad k = 1, \ldots, n}$$

[1] Recall finite geometric sum: $1 + a + \cdots + a^{n-1} = \frac{1-a^n}{1-a}$ (B-H A.8).

Winning and losing probability

Given

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}}$$

swapping $k \leftrightarrow n - k$ and $p \leftrightarrow q$ gives:

$$\boxed{Q_k = \frac{1 - \left(\frac{p}{q}\right)^{n-k}}{1 - \left(\frac{p}{q}\right)^n}}$$

Winning and losing probability

Given

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}}$$

swapping $k \leftrightarrow n - k$ and $p \leftrightarrow q$ gives:

$$\boxed{Q_k = \frac{1 - \left(\frac{p}{q}\right)^{n-k}}{1 - \left(\frac{p}{q}\right)^n}}$$

and

$$P(\text{game never ends}) = 1 - P_k - Q_k = 0$$

Summary

- For $p = \frac{1}{2}$:

$$\boxed{P_k = \frac{k}{n}, \quad k = 1, \ldots, n}$$

- For $p \neq \frac{1}{2}$

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}, \quad k = 1, \ldots, n}$$

How to reconcile these two results?

Summary

- For $p = \frac{1}{2}$:

$$\boxed{P_k = \frac{k}{n}, \quad k = 1, \dots, n}$$

- For $p \neq \frac{1}{2}$

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}, \quad k = 1, \dots, n}$$

How to reconcile these two results? Take the limit of $p \to \frac{1}{2}$.

Summary

- For $p = \frac{1}{2}$:

$$\boxed{P_k = \frac{k}{n}, \quad k = 1, \ldots, n}$$

- For $p \neq \frac{1}{2}$

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}, \quad k = 1, \ldots, n}$$

How to reconcile these two results? Take the limit of $p \to \frac{1}{2}$.

- Note that

$$P_k = \frac{\left(1 - \frac{q}{p}\right)\left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{k-1}\right)}{\left(1 - \frac{q}{p}\right)\left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{n-1}\right)}$$

Summary

- For $p = \frac{1}{2}$:

$$\boxed{P_k = \frac{k}{n}, \quad k = 1, \ldots, n}$$

- For $p \neq \frac{1}{2}$

$$\boxed{P_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n}, \quad k = 1, \ldots, n}$$

How to reconcile these two results? Take the limit of $p \to \frac{1}{2}$.

- Note that

$$P_k = \frac{\cancel{\left(1 - \frac{q}{p}\right)} \left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{k-1}\right)}{\cancel{\left(1 - \frac{q}{p}\right)} \left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{n-1}\right)} \overset{p=q}{=} \frac{k}{n}$$

Special case: $k = 3, n - k = 7$

Two gamblers, with a bankroll of $\$3$ and $\$7$, bet on independent tosses of a fair coin. The first gambler wins $\$1$ if a toss is head, with probability p; the second wins $\$1$ if a toss is a tail, with probability $q = 1 - p$. The game stops if either runs out of money. What is the probability that the first gambler wins all the money?

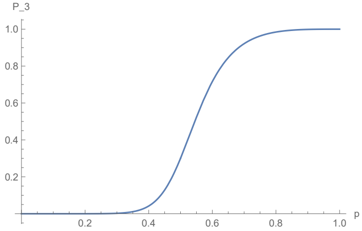
Special case: $k = 3, n - k = 7$

Two gamblers, with a bankroll of $\$3$ and $\$7$, bet on independent tosses of a fair coin. The first gambler wins $\$1$ if a toss is head, with probability p; the second wins $\$1$ if a toss is a tail, with probability $q = 1 - p$. The game stops if either runs out of money. What is the probability that the first gambler wins all the money?

- When $p = 0.56$

$$P_3 = \frac{1 - \left(\frac{0.44}{0.56}\right)^3}{1 - \left(\frac{0.44}{0.56}\right)^{10}} \approx 0.566 > 0.3$$

- P_3 vs p:



Special case: Tennis ($k = n - k = 2$)

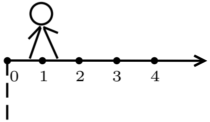
(Lecture 4) Alice plays tennis against Bob. The game is at deuce. Suppose

- Alice wins each point with probability p and loses with probability $q = 1 - p$
- Each point is played independently
- The game is won by the player who leads by 2 points

$$P_2 = P(\text{Alice eventually wins}) = \frac{1 - \left(\frac{q}{p}\right)^2}{1 - \left(\frac{q}{p}\right)^4} = \frac{1}{1 + \left(\frac{q}{p}\right)^2} = \frac{p^2}{p^2 + q^2}$$

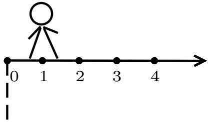
Special case: Cliff ($n = \infty$)

(PSet 3) A drunkard is standing one step away from the cliff on his left. He moves randomly, one step at a time and independently, either to the right (away from the cliff) with probability p or left (toward the cliff) with probability $q = 1 - p$.



Special case: Cliff ($n = \infty$)

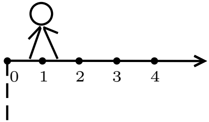
(PSet 3) A drunkard is standing one step away from the cliff on his left. He moves randomly, one step at a time and independently, either to the right (away from the cliff) with probability p or left (toward the cliff) with probability $q = 1 - p$.



$$P_1 = P(\text{never fall starting at } 1) = \lim_{n\to\infty} \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^n} = \begin{cases} 0 & p \leq \frac{1}{2} \\ 1 - \frac{q}{p} & p > \frac{1}{2} \end{cases}$$

Special case: Cliff ($n = \infty$)

(PSet 3) A drunkard is standing one step away from the cliff on his left. He moves randomly, one step at a time and independently, either to the right (away from the cliff) with probability p or left (toward the cliff) with probability $q = 1 - p$.



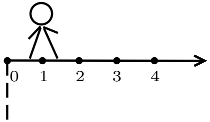
$$P_1 = P(\text{never fall starting at } 1) = \lim_{n\to\infty} \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^n} = \begin{cases} 0 & p \leq \frac{1}{2} \\ 1 - \frac{q}{p} & p > \frac{1}{2} \end{cases}$$

Similarly,

$$P_k = P(\text{never fall starting at } k) = \lim_{n\to\infty} \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n} = \begin{cases} 0 & p \leq \frac{1}{2} \\ 1 - \left(\frac{q}{p}\right)^k & p > \frac{1}{2} \end{cases}$$

Special case: Cliff ($n = \infty$)

(PSet 3) A drunkard is standing one step away from the cliff on his left. He moves randomly, one step at a time and independently, either to the right (away from the cliff) with probability p or left (toward the cliff) with probability $q = 1 - p$.



$$P_1 = P(\text{never fall starting at } 1) = \lim_{n\to\infty} \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^n} = \begin{cases} 0 & p \le \frac{1}{2} \\ 1 - \frac{q}{p} & p > \frac{1}{2} \end{cases}$$

Similarly,

$$P_k = P(\text{never fall starting at } k) = \lim_{n\to\infty} \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^n} = \begin{cases} 0 & p \le \frac{1}{2} \\ 1 - \left(\frac{q}{p}\right)^k & p > \frac{1}{2} \end{cases}$$

(This is simply because $1 - P_k = (1 - P_1)^k$)

Special case: Cliff ($n = \infty$)

$$P(\text{eventually fall starting at } k) = \begin{cases} 1 & p \le \frac{1}{2} \\ \left(\frac{q}{p}\right)^k & p > \frac{1}{2} \end{cases}$$