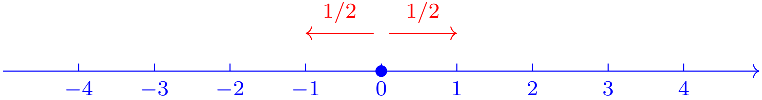
S&DS 241 Lecture 11 (optional)

Random walk: Probability of eventual return

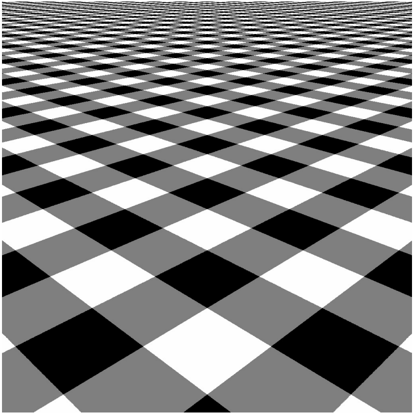
Drunkard's Walks in one dimension

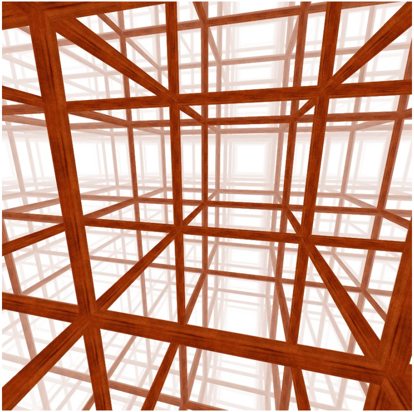
A drunkard walks randomly in an idealized 1-dimensional city.[1] The city is infinite and arranged in 1-dimensional equally-spaced grid, and at every point, the drunkard chooses one of the 2 possible routes (including the one he came from) with equal probability. Formally, this is a symmetric random walk on the set of integers.



[1] For instance, the drunkard wanders into a very long alley.

Drunkard's Walks in two or three dimensions





Drunkard's Walk

Question

Will the drunkard ever return to the starting point?

Drunkard's Walk

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Will the drunkard ever return to the starting point?

Answer

Always return in 1 or 2 dimensional space, but not necessarily in 3 and higher dimensional space.

> "One may summarize these results by stating that one should not get drunk in more than two dimensions."
>
> — Grinstead-Snell, p. 478

Let's start with one dimension

Two methods

$$P(\text{eventual return}) = 1$$

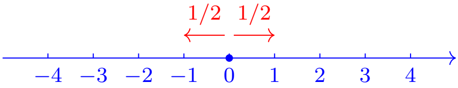
Method 1 Direct calculation

Method 2 Proof by contradiction (indirect but easy to extend to higher dimensions)

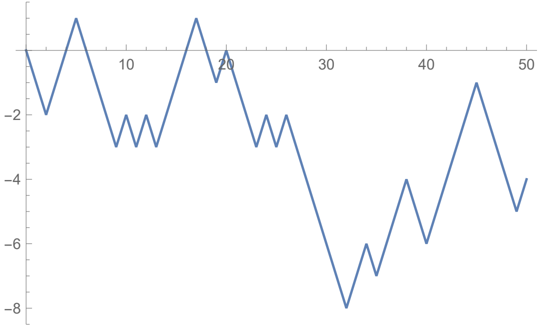
Method 1: Direct calculation

Symmetric random walk

A particle starts at 0, and at each step it either moves 1 unit to the right with probability $1/2$ or to the left with probability $1/2$, independently.



Let S_n be the particle's position after n steps.



PMF of S_n

Let

$$X_i = i^{\text{th}} \text{ step} = \begin{cases} +1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2 \end{cases}$$

Then

$$S_n = \underbrace{X_1 + \cdots + X_n}_{\text{iid}}$$

PMF of S_n

Let

$$X_i = i^{\text{th}} \text{ step} = \begin{cases} +1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2 \end{cases}$$

Then

$$S_n = \underbrace{X_1 + \cdots + X_n}_{\text{iid}}$$

Alternatively,

- Let X= number of steps to the right $\sim \text{Bin}(n, 1/2)$
- Then $S_n = 2X - n \in \{-n, -n+2, \ldots, n-2, n\}$ and

$$P(S_n = j) = P(X = (n+j)/2) = \binom{n}{\frac{n+j}{2}} 2^{-n}$$

Probability of eventual return

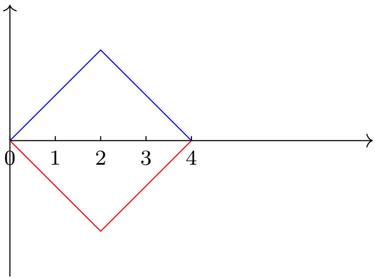
- Possible return time: $2, 4, 6, \ldots$ (all even numbers)
- According to the time of first return,

$$
\begin{aligned}
& P(\text{eventual return}) \\
= \; & P(\text{first return at time 2}) + P(\text{first return at time 4}) + \cdots \\
= \; & \sum_{n \geq 1} P(\text{first return at time } 2n)
\end{aligned}
$$

First return at time 2

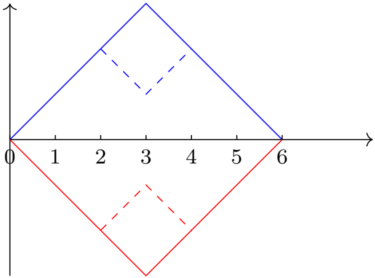
$$\begin{aligned} P(\text{first return at time } 2) &= P(S_2 = 0) \\ &= P(\texttt{+-}) + P(\texttt{-+}) = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} \end{aligned}$$

First return at time 4



$$P(\text{first return at time } 4) = P(S_2 \neq 0, S_4 = 0)$$
$$= P(\texttt{++--}) + P(\texttt{--++}) = 2 \times \frac{1}{2^4} = \frac{1}{8}$$

First return at time 6



$$
\begin{aligned}
& P(\text{first return at time } 6) \\
= \ & P(S_2 \neq 0, S_4 \neq 0, S_6 = 0) \\
= \ & P(\texttt{+++---}) + P(\texttt{++-+--}) + P(\texttt{---+++}) + P(\texttt{--+-++}) \\
= \ & 4 \times \frac{1}{2^6} = \frac{1}{16}
\end{aligned}
$$

First return at time $2n$

More generally:

$$P(\text{first return at time } 2n) = \frac{\text{number of paths } (0,0) \rightsquigarrow (2n,0) \text{ that stay above or below horizontal axis}}{2^{2n}}$$

First return at time $2n$

$P(\text{first return at time } 2n)$

First return at time $2n$

$$
\begin{aligned}
& P(\text{first return at time } 2n) \\
= \; & P(\text{never return to 0 before } 2n, \text{ return to 0 at } 2n)
\end{aligned}
$$

First return at time $2n$

$$
\begin{aligned}
& P(\text{first return at time } 2n) \\
= \; & P(\text{never return to 0 before } 2n\text{, return to 0 at } 2n) \\
= \; & P(\text{return to 0 at } 2n) - P(\text{return to 0 at } 2n \text{ but not for the first time})
\end{aligned}
$$

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= \ & \underbrace{P(S_{2n} = 0)}_{\binom{2n}{n}2^{-2n}} - P(S_t = 0 \text{ for some } 2 \leq t \leq 2n-2, S_{2n} = 0)
\end{aligned}
$$

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\end{aligned}
$$

By definition, $S_1 = \pm 1$, $S_{2n-1} = \pm 1$ — four possibilities.

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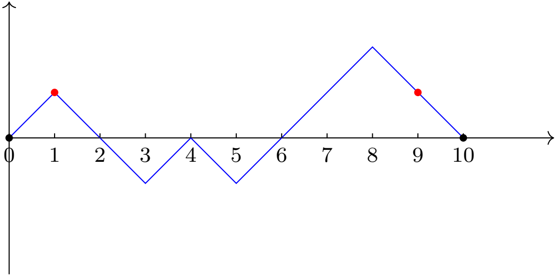
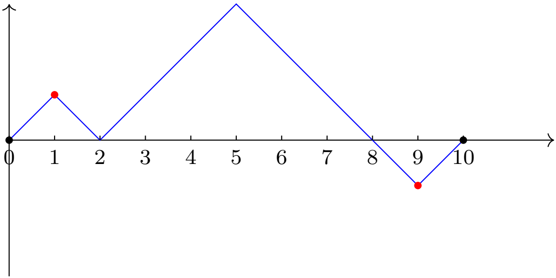


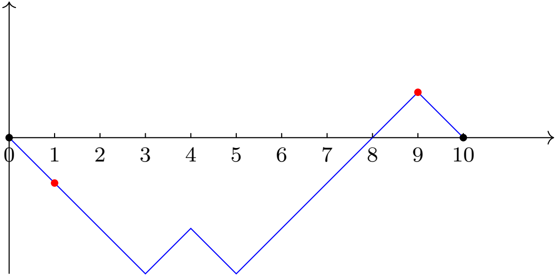
$$S_0 = 0, S_1 = 1, \ldots, S_{2n-1} = 1, S_{2n} = 0$$

$+-$

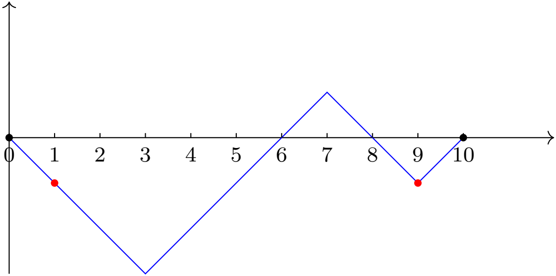


$$S_0 = 0, S_1 = 1, \ldots, S_{2n-1} = -1, S_{2n} = 0$$

$-+$



$$S_0 = 0, S_1 = -1, \ldots, S_{2n-1} = 1, S_{2n} = 0$$



$$S_0 = 0, S_1 = -1, \ldots, S_{2n-1} = -1, S_{2n} = 0$$

Four scenarios

$$
\begin{aligned}
&P(S_t = 0 \text{ for some } 2 \le t \le 2n-2, S_{2n} = 0) \\
=&P(S_1 = \mathbf{+1}, S_t = 0 \text{ for some } 2 \le t \le 2n-2, S_{2n-1} = \mathbf{+1}, S_{2n} = 0) \\
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\end{aligned}
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We shall show that all four probabilities are equal to

$$\binom{2n-2}{n} 2^{-2n}$$

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\end{aligned}
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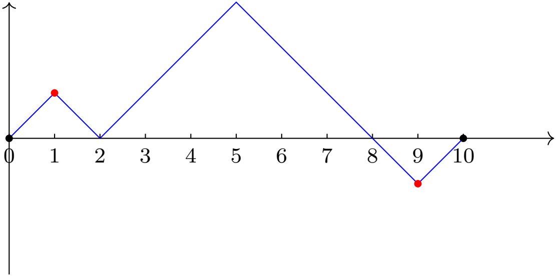
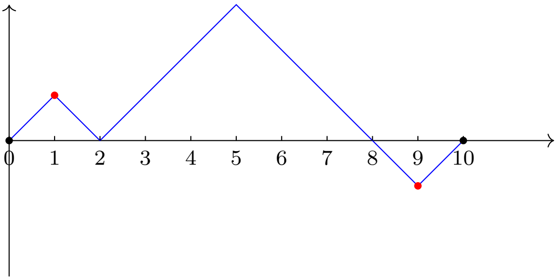


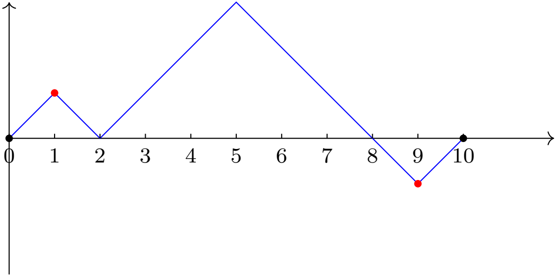
$$P(S_1 = +1, S_t = 0 \text{ for some } 2 \le t \le 2n-2, S_{2n-1} = -1, S_{2n} = 0)$$

$+-$



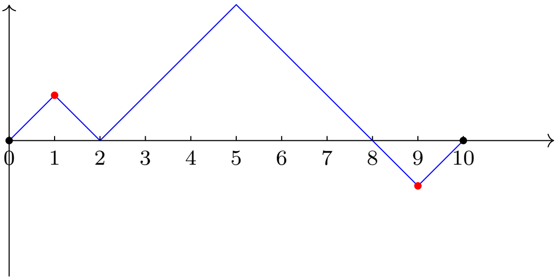
$$P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n-2, S_{2n-1} = -1, S_{2n} = 0)$$
$$= P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0)$$

$+-$



$$
\begin{aligned}
&P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n-2, S_{2n-1} = -1, S_{2n} = 0) \\
=&P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0) \\
=&P(\text{first step=+}, \underbrace{\text{move from } +1 \text{ to } -1 \text{ in } 2n-2 \text{ steps}}_{n \text{ ``–'' out of } 2n-2 \text{ steps}}, \text{ last step=+})
\end{aligned}
$$

$+-$



$$P(S_1 = +1, S_t = 0 \text{ for some } 2 \le t \le 2n-2, S_{2n-1} = -1, S_{2n} = 0)$$

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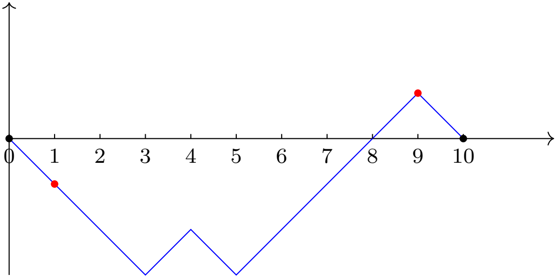
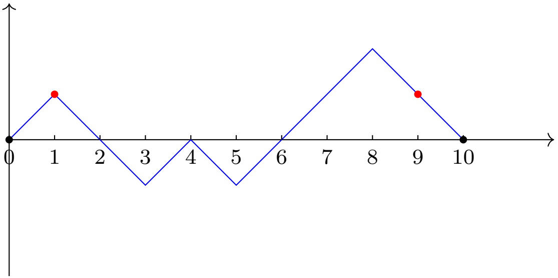
$$= \frac{1}{2} \times \binom{2n-2}{n} 2^{-(2n-2)} \times \frac{1}{2} = \binom{2n-2}{n} 2^{-2n}$$

$-+$



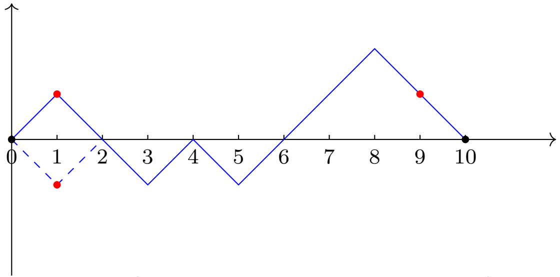
$$
\begin{aligned}
& P(S_1 = -1, S_t = 0 \text{ for some } 2 \le t \le 2n-2, S_{2n-1} = +1, S_{2n} = 0) \\
=& P(S_1 = -1, S_{2n-1} = +1, S_{2n} = 0) \\
=& P(\text{first step=}\textbf{-}, \underbrace{\text{move from } -1 \text{ to } +1 \text{ in } 2n-2 \text{ steps}}_{n \text{ ``+'' out of } 2n-2 \text{ steps}}, \text{ last step=}\textbf{-}) \\
=& \binom{2n-2}{n} 2^{-2n}
\end{aligned}
$$

$++$



0
1
2
3
4
5
6
7
8
9
10

++



- Reflection principle (reflecting at the 1st crossing): For every path $(1,1) \rightsquigarrow (2n-1,1)$ that crosses the horizontal axis, there is another path $(1,-1) \rightsquigarrow (2n-1,1)$, and vice versa

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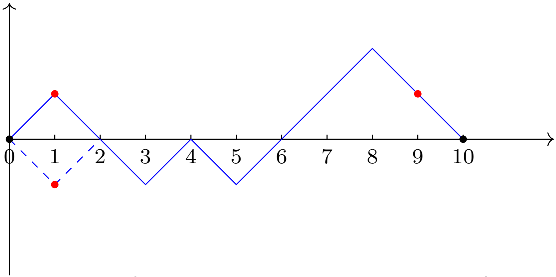
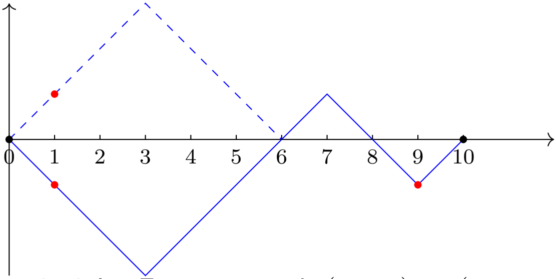
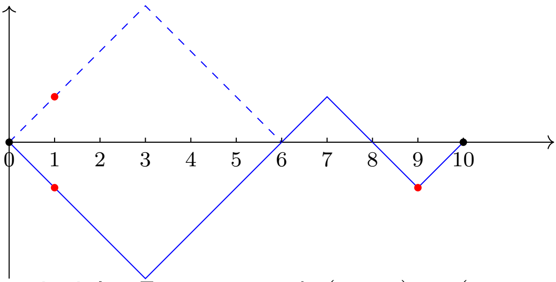


- Reflection principle (reflecting at the 1st crossing): For every path $(1,1) \rightsquigarrow (2n-1,1)$ that crosses the horizontal axis, there is another path $(1,-1) \rightsquigarrow (2n-1,1)$, and vice versa
- Therefore

$$
\begin{aligned}
&P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n-2, S_{2n-1} = +1, S_{2n} = 0) \\
=&P(S_1 = -1, S_{2n-1} = +1, S_{2n} = 0) \\
=&\binom{2n-2}{n} 2^{-2n}
\end{aligned}
$$



- <u>Reflection principle</u>: For every path $(1, -1) \rightsquigarrow (2n-1, -1)$ that crosses the horizontal axis, there is another path $(1, 1) \rightsquigarrow (2n-1, -1)$, and vice versa



- Reflection principle: For every path $(1,-1) \rightsquigarrow (2n-1,-1)$ that crosses the horizontal axis, there is another path $(1,1) \rightsquigarrow (2n-1,-1)$, and vice versa
- Therefore

$$\begin{aligned} & P(S_1 = -1, S_t = 0 \text{ for some } 2 \le t \le 2n-2, S_{2n-1} = -1, S_{2n} = 0) \\ = & P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0) \\ = & \binom{2n-2}{n} 2^{-2n} \end{aligned}$$

Put everything together

$$P(\text{return to 0 at } 2n) = \binom{2n}{n} 2^{-2n}$$

$$P(\text{return to 0 at } 2n \text{ but not for the first time}) = 4 \times \binom{2n-2}{n} 2^{-2n}$$

So

$$\begin{aligned} & P(\text{return to 0 at } 2n \text{ for the first time}) \\ = & \binom{2n}{n} 2^{-2n} - 4\binom{2n-2}{n} 2^{-2n} = \left(1 - 4\frac{n(n-1)}{2n(2n-1)}\right)\binom{2n}{n} 2^{-2n} \\ = & \frac{1}{2n-1}\binom{2n}{n} 2^{-2n} \end{aligned}$$

First few values: $\frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{5}{128}, \frac{7}{256}, \ldots$

Probability of eventual return

$$P(\text{eventual return}) = \sum_{n\geq 1} P(\text{first return at time } 2n)$$
$$= \sum_{n\geq 1} \frac{1}{2n-1}\binom{2n}{n} 2^{-2n} = \mathbf{1}$$

using Taylor expansion

$$1-\sqrt{1-4x} = \sum_{n\geq 1} \frac{1}{2n-1}\binom{2n}{n} x^n$$

with $x = 1/4$

Method 2: Proof by contradiction

Returning finitely often

- Let $P_0 = P(\text{never returns to } 0)$

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$$E = \{\text{drunkard returns to } 0 \ \underline{\text{finitely often}}\}$$

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- Here finitely often means coming back finitely many times, e.g., 0 time, 1 time, 2 times, $\cdots$. Thus $P_0 \leq P(E)$.
- **Question**: How to express $P(E)$ using P_0?

According to time of the last return

$$
\begin{aligned}
E &= \{\text{returns finitely often}\} \\
&= \{\text{never turns}\} \\
&\quad \cup \{\text{returns at time } 2\text{, then never returns}\} \\
&\quad \cup \{\text{returns at time } 4\text{, then never returns}\} \\
&\quad \cup \cdots
\end{aligned}
$$

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$$\begin{aligned}
E &= \{\text{returns finitely often}\} \\
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&\quad \cup \{\text{returns at time } 4\text{, then never returns}\} \\
&\quad \cup \cdots
\end{aligned}$$

The union is over mutually exclusive events. Thus:

$$\begin{aligned}
P(E) &= P(\text{never turns}) \\
&\qquad + P(\text{returns at time } 2\text{, then never returns}) \\
&\qquad + P(\text{returns at time } 4\text{, then never returns}) \\
&\qquad + \cdots \\
&= P_0 + P(S_2 = 0)P_0 + P(S_4 = 0)P_0 + \cdots
\end{aligned}$$

Therefore

$$P\left(E\right) = \left[1 + \sum_{n=1}^{\infty} P\left(S_{2n} = 0\right)\right] \cdot P_0,$$

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$$P(E) = \left[1 + \underbrace{\sum_{n=1}^{\infty} P(S_{2n} = 0)}_{=+\infty \text{ (next slide)}}\right] \cdot P_0,$$

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$$P(E) = \left[1 + \underbrace{\sum_{n=1}^{\infty} P(S_{2n} = 0)}_{=+\infty \text{ (next slide)}}\right] \cdot P_0,$$

Suppose $P_0 \neq 0$. Then $P(E) = +\infty$. Contradiction! Thus it must be

$$P_0 = 0$$

Stirling Approximation

- Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

where LHS $\sim$ RHS means $\frac{\text{LHS}}{\text{RHS}} \to 1$ as $n \to \infty$.

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- Hence

$$P\left(S_{2n}=0\right)=\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n} \overset{\text{Stirling}}{\sim} \frac{\sqrt{2\pi\cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}\left(\frac{1}{2}\right)^{2n}=\frac{1}{\sqrt{\pi n}}$$

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- Hence

$$P\left(S_{2n}=0\right) = \frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n} \overset{\text{Stirling}}{\sim} \frac{\sqrt{2\pi \cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}\left(\frac{1}{2}\right)^{2n} = \frac{1}{\sqrt{\pi n}}$$

- Recall an important fact from calculus:

$$\sum_{n\geq 1}\frac{1}{n^a} = \begin{cases}\infty & a \leq 1\\ \text{finite} & a > 1\end{cases}$$

Stirling Approximation

- Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

where LHS $\sim$ RHS means $\frac{\text{LHS}}{\text{RHS}} \to 1$ as $n \to \infty$.

- Hence

$$P\left(S_{2n}=0\right) = \frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n} \overset{\text{Stirling}}{\sim} \frac{\sqrt{2\pi \cdot 2n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2}\left(\frac{1}{2}\right)^{2n} = \frac{1}{\sqrt{\pi n}}$$

- Recall an important fact from calculus:

$$\sum_{n\geq 1}\frac{1}{n^a} = \begin{cases}\infty & a \leq 1\\ \text{finite} & a > 1\end{cases}$$

- Hence

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{\pi n}} = \infty \implies \sum_{n=1}^{\infty} P\left(S_{2n}=0\right) = \infty$$

Conclusion: 1-D random walk always comes back

$$\begin{aligned} P_0 &= P\,(\text{the drunkard never returns to } 0) = 0, \\ P\,(E) &= P\,(\text{the drunkard returns to } 0 \text{ finitely often}) = 0. \end{aligned}$$

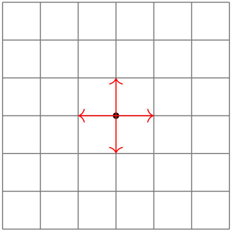
i.e.,

$$\begin{aligned} P\,(\text{the drunkard returns to } 0 \text{ eventually}) &= 1, \\ P\,(\text{the drunkard returns to } 0 \text{ infinitely often}) &= 1. \end{aligned}$$

2 and 3-dim random walks

Drunkard's Walks on the plane

A drunkard walks randomly in an idealized 2-dimensional city. The city is infinite and arranged in an equally-spaced square grid. At every intersection, the drunkard chooses one of the 4 directions: N/S/E/W, with equal probability. Formally, this is a random walk on $\mathbb{Z}^2$.



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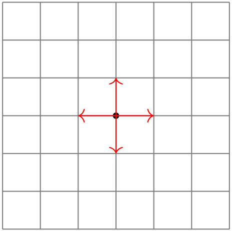
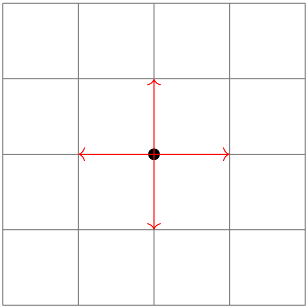


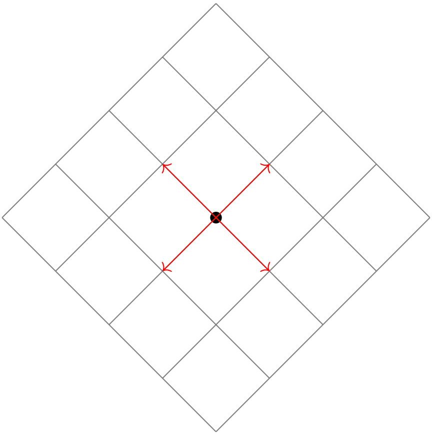
Question

What is the probability of eventual return?

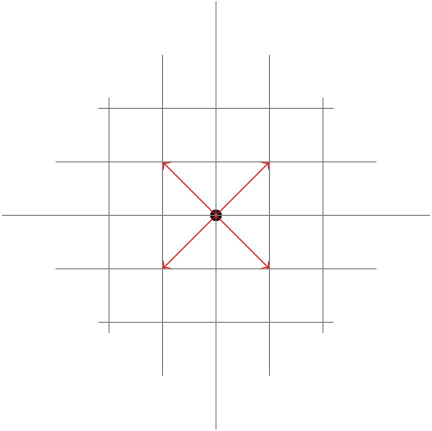
Equivalent view



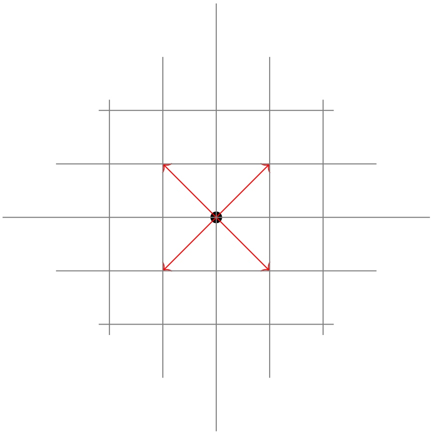
Equivalent view



Equivalent view



Equivalent view



We can consider the walk which moves to NE/NW/SW/SW:

$$(x, y) \to (x \pm 1, y \pm 1)$$

2 Dimensional Random Walk

- Two independent sequences of independent random variables:

$$P(X_i = 1) = P(X_i = -1) = 1/2 \qquad \text{horizontal steps}$$
$$P(Y_i = 1) = P(Y_i = -1) = 1/2 \qquad \text{vertical steps}$$

- Position at time n:

$$S_n = \sum_{i=1}^{n} X_i \qquad \text{horizontal coordinate}$$
$$T_n = \sum_{i=1}^{n} Y_i \qquad \text{vertical coordinate}$$

Apply the same reasoning

- Let $P_0 = P(\text{never returns})$.

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- Key identity is the same as in one dimension:

$$P(E) = \sum_{n=0}^{\infty} \underbrace{P(S_{2n} = 0, T_{2n} = 0) \times P_0}_{P(\text{returns at time } 2n, \text{ then never returns})}$$

It boils down to $\sum_{n \geq 0} P(S_{2n} = 0, T_{2n} = 0) \stackrel{?}{=} \infty$

Using Stirling's formula again

$$P\left(S_{2n}=0, T_{2n}=0\right)=P\left(S_{2n}=0\right)P\left(T_{2n}=0\right)\sim\left(\frac{1}{\sqrt{\pi n}}\right)^2$$

Important facts:

- $P(S_{2n}=0)\sim\frac{1}{\sqrt{\pi n}}$
- $\sum\limits_{n=1}^{\infty} 1/n=+\infty!$

If $P_0\neq 0$, then $P\left(E\right)=+\infty$. Contradiction! Thus $P_0=0$

Conclusion: 2-D random walk always comes back

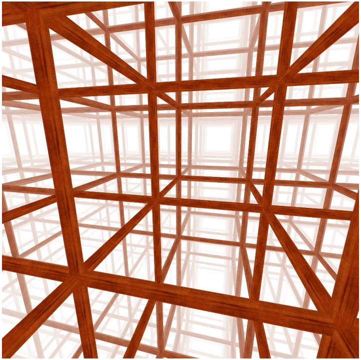
$$\begin{aligned} P_0 &= P\,(\text{the drunkard never returns }) = 0, \\ P\,(E) &= P\,(\text{the drunkard returns finitely often}) = 0. \end{aligned}$$

i.e.,

$$\begin{aligned} P\,(\text{the drunkard returns eventually}) &= 1, \\ P\,(\text{the drunkard returns infinitely often}) &= 1. \end{aligned}$$

Drunkard's Walks in space

A drunkard walks randomly in an idealized 3-dimensional city. The city is infinite and arranged in an equally-spaced cubic grid. At every intersection, the drunkard chooses one of the 6 directions: up/down/left/right/back/forth, with equal probability. Formally, this is a random walk on $\mathbb{Z}^3$.



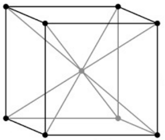
(Simplified) 3-dimensional Walk

Three independent sequences of independent random variables:

$$
\begin{aligned}
P(X_i = 1) &= P(X_i = -1) = 1/2, \\
P(Y_i = 1) &= P(Y_i = -1) = 1/2, \\
P(Z_i = 1) &= P(Z_i = -1) = 1/2.
\end{aligned}
$$

Position at time n:

$$S_n = \sum_{i=1}^{n} X_i,\ T_n = \sum_{i=1}^{n} Y_i,\ U_n = \sum_{i=1}^{n} Z_i.$$



8 directions: $(x, y, z) \to (x \pm 1, y \pm 1, z \pm 1)$

Key Difference

- The same reasoning leads to

$$P(E) = \underbrace{\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0)}_{\text{but now this is finite!}} \cdot P_0,$$

since

$$\begin{aligned} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) =& P(S_{2n} = 0) P(T_{2n} = 0) P(U_{2n} = 0) \\ &\sim \left(\frac{1}{\sqrt{\pi n}}\right)^3 \end{aligned}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}} \approx 2.6 < \infty$

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and $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}} \approx 2.6 < \infty$

- Thus it is not immediately clear whether $P_0 = 0$ or not.

In fact: $P_0 > 0$

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- Let $X =$ number of returns. Then

$$X = \sum_{n \geq 0} \mathbf{1}_{\{S_{2n}=0, T_{2n}=0, U_{2n}=0\}}$$

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- Let $X =$ number of returns. Then

$$X = \sum_{n \geq 0} \mathbf{1}_{\{S_{2n}=0, T_{2n}=0, U_{2n}=0\}}$$

- We know $X = +\infty$ with probability 1, but

$$E(X) = \sum_{n=0}^{\infty} P\left(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0\right) < +\infty$$

contradition!

Conclusion: 3-D random walk might not return

$$P(\text{never return}) > 0^{\dagger}$$

$^{\dagger}P(\text{never return}) \approx 72\%$

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$$P(\text{never return}) > 0^{\dagger}$$

The same holds for

- the original 3-D walk with 6 directions (Grinstead-Snell, Sec 12.1 Ex 14): $P(\text{never return}) \approx 66\%$
- walks in higher dimensions

George Pólya

† $P(\text{never return}) \approx 72\%$

Summary

What we have learned: a dichotomy

$$\sum_{k\geq 0} P(\text{return at time } k) = \infty \Leftrightarrow P(\text{never return}) = 0$$
$$\sum_{k\geq 0} P(\text{return at time } k) < \infty \Leftrightarrow P(\text{never return}) > 0$$

This is applicable to analyzing other walks, e.g., asymmetric ones (HW)

More precisely

- If $\sum_{k\geq 0} P(\text{return at time } k) = \infty$, then

$$P(\text{return finitely often}) = 0$$
$$P(\text{never return}) = 0$$

- If $\sum_{k\geq 0} P(\text{return at time } k) < \infty$, since $EX < \infty$, X is finite with probability one, then

$$P(\text{return finitely often}) = 1$$
$$P(\text{never return}) = \frac{1}{\sum_{k\geq 0} P(\text{return at time } k)}$$