S&DS 241 Lecture 11 (optional)

Random walk: Probability of eventual return
A drunkard walks randomly in an idealized 1-dimensional city. The city is infinite and arranged in 1-dimensional equally-spaced grid, and at every point, the drunkard chooses one of the 2 possible routes (including the one he came from) with equal probability. Formally, this is a symmetric random walk on the set of integers.

\[ \frac{1}{2} \quad \frac{1}{2} \]

\[ -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

\footnote{For instance, the drunkard wanders into a very long alley.}
Drunkard’s Walks in two or three dimensions
Drunkard’s Walk

Question
Will the drunkard ever return to the starting point?
## Drunkard’s Walk

### Question
Will the drunkard ever return to the starting point?

### Answer
Always return in 1 or 2 dimensional space, but not necessarily in 3 and higher dimensional space.
“One may summarize these results by stating that one should not get drunk in more than two dimensions.”

— Grinstead-Snell, p. 478
Let’s start with one dimension
Two methods

\[ P(\text{eventual return}) = 1 \]

**Method 1**  Direct calculation

**Method 2**  Proof by contradiction (indirect but easy to extend to higher dimensions)
Method 1: Direct calculation
Symmetric random walk

A particle starts at 0, and at each step it either moves 1 unit to the right with probability 1/2 or to the left with probability 1/2, independently.

Let $S_n$ be the particle’s position after $n$ steps.
PMF of $S_n$

Let

$$X_i = \text{ith step} = \begin{cases} +1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2 \end{cases}$$

Then

$$S_n = X_1 + \cdots + X_n$$
PMF of $S_n$

Let

$$X_i = \text{ith step} = \begin{cases} +1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2 \end{cases}$$

Then

$$S_n = X_1 + \cdots + X_n \underbrace{\text{iid}}$$

Alternatively,

- Let $X = \text{number of steps to the right} \sim \text{Bin}(n, 1/2)$
- Then $S_n = 2X - n \in \{-n, -n + 2, \ldots, n - 2, n\}$ and

$$P(S_n = j) = P(X = (n + j)/2) = \left( \frac{n}{n+j} \right) 2^{-n}$$
Probability of eventual return

- Possible return time: 2, 4, 6, ... (all even numbers)
- According to the time of first return,

\[
P(\text{eventual return})
= P(\text{first return at time 2}) + P(\text{first return at time 4}) + \cdots
= \sum_{n \geq 1} P(\text{first return at time } 2n)
\]
First return at time 2

\[ P(\text{first return at time 2}) = P(S_2 = 0) \]

\[ = P(+-) + P(-+) = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} \]
First return at time 4

\[ P(\text{first return at time 4}) = P(S_2 \neq 0, S_4 = 0) \]

\[ = P(++++) + P(----) = 2 \times \frac{1}{2^4} = \frac{1}{8} \]
First return at time 6

\[ P(\text{first return at time 6}) \]
\[ = P(S_2 \neq 0, S_4 \neq 0, S_6 = 0) \]
\[ = P(++++--) + P(+-+-++) + P(--+-++) + P(---+++) \]
\[ = 4 \times \frac{1}{2^6} = \frac{1}{16} \]
First return at time $2n$

More generally:

$$P(\text{first return at time } 2n) = \frac{\text{number of paths } (0, 0) \rightsquigarrow (2n, 0) \text{ that stay above or below horizontal axis}}{2^{2n}}$$
First return at time $2n$

$$P(\text{first return at time } 2n)$$
First return at time $2n$

\[ P(\text{first return at time } 2n) = P(\text{never return to 0 before } 2n, \text{ return to 0 at } 2n) \]
First return at time $2n$

\[
P(\text{first return at time } 2n) = P(\text{never return to 0 before } 2n, \text{ return to 0 at } 2n) - P(\text{return to 0 at } 2n \text{ but not for the first time})\]
First return at time $2n$

\[
P(\text{first return at time } 2n)
= P(\text{never return to 0 before } 2n, \text{ return to 0 at } 2n)
= P(\text{return to 0 at } 2n) - P(\text{return to 0 at } 2n \text{ but not for the first time})
= P(S_{2n} = 0) - P(S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n} = 0)
\]

By definition, $S_1 = \pm 1, S_{2n-1} = \pm 1$— four possibilities.
First return at time $2n$

\[
P(\text{first return at time } 2n) = P(\text{never return to 0 before } 2n, \text{ return to 0 at } 2n)
\]
\[
= P(\text{return to 0 at } 2n) - P(\text{return to 0 at } 2n \text{ but not for the first time})
\]
\[
= P(S_{2n} = 0) - P(S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n} = 0)
\]

By definition, $S_1 = \pm 1$, $S_{2n-1} = \pm 1$ — four possibilities.
$S_0 = 0, S_1 = 1, \ldots, S_{2n-1} = 1, S_{2n} = 0$
$S_0 = 0, S_1 = 1, \ldots, S_{2n-1} = -1, S_{2n} = 0$
$S_0 = 0, S_1 = -1, \ldots, S_{2n-1} = 1, S_{2n} = 0$
$S_0 = 0, S_1 = -1, \ldots, S_{2n-1} = -1, S_{2n} = 0$
Four scenarios

\[ P(S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n} = 0) \]

\[ = P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \]

\[ + P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]

\[ + P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \]

\[ + P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]
Four scenarios

\[ P(S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n} = 0) \]
\[ = P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \]
\[ + P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]
\[ + P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \]
\[ + P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]

We shall show that all four probabilities are equal to

\[ \left( \frac{2n - 2}{n} \right) 2^{-2n} \]
Four scenarios

\[ P(S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n} = 0) \]
\[ = P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \]
\[ + P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]
\[ + P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \]
\[ + P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]

We shall show that all four probabilities are equal to

\[ \binom{2n - 2}{n} 2^{-2n} \]
\[ P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]
\[ P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]

\[ = P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0) \]
\[ P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]
\[ = P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0) \]
\[ = P(\text{first step} = +, \text{ move from } +1 \text{ to } -1 \text{ in } 2n - 2 \text{ steps, last step} = +) \]
\[ n \text{ "-" out of } 2n - 2 \text{ steps} \]
\[ P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) \]
\[ = P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0) \]
\[ = P(\text{first step}=+\text{, move from } +1 \text{ to } -1 \text{ in } 2n-2 \text{ steps, last step}=+\text{)} \]
\[ = \frac{1}{2} \times \binom{2n-2}{n} 2^{-(2n-2)} \times \frac{1}{2} = \binom{2n-2}{n} 2^{-2n} \]
\[
P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0) \\
= P(S_1 = -1, S_{2n-1} = +1, S_{2n} = 0) \\
= P(\text{first step}=-, \text{ move from } -1 \text{ to } +1 \text{ in } 2n - 2 \text{ steps, last step}=--) \\
= \binom{2n-2}{n} 2^{-2n}
\]
• Reflection principle (reflecting at the 1st crossing): For every path \((1, 1) \rightarrow (2^n - 1, 1)\) that crosses the horizontal axis, there is another path \((1, -1) \rightarrow (2^n - 1, 1)\), and vice versa.

• Therefore \(P(S_1 = +1, S_{2^n} = 0) = P(S_1 = -1, S_{2^n - 1} = +1, S_{2^n} = 0)\).
Reflection principle (reflecting at the 1st crossing): For every path \((1,1) \rightsquigarrow (2n-1,1)\) that crosses the horizontal axis, there is another path \((1,-1) \rightsquigarrow (2n-1,1)\), and vice versa.
• Reflection principle (reflecting at the 1st crossing): For every path 
$$(1, 1) \sim (2n - 1, 1)$$ that crosses the horizontal axis, there is another 
path $$(1, -1) \sim (2n - 1, 1)$$, and vice versa

• Therefore

$$P(S_1 = +1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = +1, S_{2n} = 0)$$
$$= P(S_1 = -1, S_{2n-1} = +1, S_{2n} = 0)$$
$$= \binom{2n - 2}{n} 2^{-2n}$$
• Reflection principle: For every path \((1, -1) \rightsquigarrow (2n - 1, -1)\) that crosses the horizontal axis, there is another path \((1, 1) \rightsquigarrow (2n - 1, -1)\), and vice versa
• Reflection principle: For every path \((1, -1) \leadsto (2n - 1, -1)\) that crosses the horizontal axis, there is another path \((1, 1) \leadsto (2n - 1, -1)\), and vice versa

• Therefore

\[
P(S_1 = -1, S_t = 0 \text{ for some } 2 \leq t \leq 2n - 2, S_{2n-1} = -1, S_{2n} = 0) = P(S_1 = +1, S_{2n-1} = -1, S_{2n} = 0)
= \left(\frac{2n - 2}{n}\right) 2^{-2n}
\]
Put everything together

\[ P(\text{return to 0 at } 2n) = \binom{2n}{n} 2^{-2n} \]

\[ P(\text{return to 0 at } 2n \text{ but not for the first time}) = 4 \times \binom{2n - 2}{n} 2^{-2n} \]

So

\[ P(\text{return to 0 at } 2n \text{ for the first time}) \]

\[ = \binom{2n}{n} 2^{-2n} - 4 \binom{2n - 2}{n} 2^{-2n} = \left( 1 - 4 \frac{n(n - 1)}{2n(2n - 1)} \right) \binom{2n}{n} 2^{-2n} \]

\[ = \frac{1}{2n - 1} \binom{2n}{n} 2^{-2n} \]

First few values: \( \frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{5}{128}, \frac{7}{256}, \ldots \)
Probability of eventual return

\[ P(\text{eventual return}) = \sum_{n \geq 1} P(\text{first return at time } 2n) = \sum_{n \geq 1} \frac{1}{2n - 1} \binom{2n}{n} 2^{-2n} = 1 \]

using Taylor expansion

\[ 1 - \sqrt{1 - 4x} = \sum_{n \geq 1} \frac{1}{2n - 1} \binom{2n}{n} x^n \]

with \( x = 1/4 \)
Method 2: Proof by contradiction
Returning finitely often

- Let $P_0 = P(\text{never returns to 0})$
Returning finitely often

- Let \( P_0 = P(\text{never returns to } 0) \)
- Define the event:

\[
E = \{ \text{drunkard returns to } 0 \text{ finitely often} \}
\]
Returning finitely often

- Let $P_0 = P(\text{never returns to 0})$
- Define the event:

$$E = \{\text{drunkard returns to 0 \underline{finitely} often}\}$$

- Here \underline{finitely often} means coming back finitely many times, e.g., 0 time, 1 time, 2 times, \cdots. Thus $P_0 \leq P(E)$. 


Returning finitely often

• Let $P_0 = P(\text{never returns to 0})$
• Define the event:

$$E = \{\text{drunkard returns to 0 finitely often}\}$$

• Here finitely often means coming back finitely many times, e.g., 0 time, 1 time, 2 times, ... Thus $P_0 \leq P(E)$.

• **Question:** How to express $P(E)$ using $P_0$?
According to time of the last return

\[ E = \{ \text{returns finitely often} \} = \{ \text{never turns} \} \cup \{ \text{returns at time 2, then never returns} \} \cup \{ \text{returns at time 4, then never returns} \} \cup \cdots \]
According to time of the last return

\[ E = \{ \text{returns finitely often} \} = \{ \text{never turns} \} \cup \{ \text{returns at time 2, then never returns} \} \cup \{ \text{returns at time 4, then never returns} \} \cup \cdots \]

The union is over mutually exclusive events. Thus:

\[
P(E) = P(\text{never turns}) + P(\text{returns at time 2, then never returns}) + P(\text{returns at time 4, then never returns}) + \cdots
\]

\[ = P_0 + P(S_2 = 0)P_0 + P(S_4 = 0)P_0 + \cdots \]
Therefore

\[ P(E) = \left[ 1 + \sum_{n=1}^{\infty} P(S_{2n} = 0) \right] \cdot P_0, \]
Therefore

\[ P(E) = \left[ 1 + \sum_{n=1}^{\infty} P(S_{2n} = 0) \right] \cdot P_0, \]

\[ = +\infty \text{ (next slide)} \]
Therefore

\[
P(E) = \left[ 1 + \sum_{n=1}^{\infty} P(S_{2n} = 0) \right] \cdot P_0,
\]

Suppose \( P_0 \neq 0 \). Then \( P(E) = +\infty \). Contradiction! Thus it must be

\[ P_0 = 0 \]
Stirling Approximation

- **Stirling’s formula:**

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \]

where LHS \( \sim \) RHS means \( \frac{\text{LHS}}{\text{RHS}} \to 1 \) as \( n \to \infty \).
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• Hence

\[
P(S_{2n} = 0) = \frac{(2n)!}{(n!)^2} \left( \frac{1}{2} \right)^{2n} \text{Stirling} \quad \frac{\sqrt{2\pi \cdot 2n} \left( \frac{2n}{e} \right)^{2n}}{(\sqrt{2\pi n} \left( \frac{n}{e} \right)^n)^2} \left( \frac{1}{2} \right)^{2n} = \frac{1}{\sqrt{\pi n}}
\]
Stirling Approximation

- **Stirling’s formula:**

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where LHS \( \sim \) RHS means \( \frac{\text{LHS}}{\text{RHS}} \rightarrow 1 \) as \( n \rightarrow \infty \).

- Hence

\[
P(S_{2n} = 0) = \frac{(2n)!}{(n!)^2} \left( \frac{1}{2} \right)^{2n} \text{Stirling} \sim \frac{\sqrt{2\pi \cdot 2n} \left( \frac{2n}{e} \right)^{2n}}{(\sqrt{2\pi n} \left( \frac{n}{e} \right)^n)^2} \left( \frac{1}{2} \right)^{2n} = \frac{1}{\sqrt{\pi n}}
\]

- Recall an important fact from calculus:

\[
\sum_{n \geq 1} \frac{1}{n^a} = \begin{cases} 
\infty & a \leq 1 \\
\text{finite} & a > 1 
\end{cases}
\]
Stirling Approximation

- **Stirling’s formula:**

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \]

where LHS \( \sim \) RHS means \( \frac{\text{LHS}}{\text{RHS}} \to 1 \) as \( n \to \infty \).

- Hence

\[ P (S_{2n} = 0) = \frac{(2n)!}{(n!)^2} \left( \frac{1}{2} \right)^{2n} \sim \frac{\sqrt{2\pi \cdot 2n \left( \frac{2n}{e} \right)^{2n}}}{(\sqrt{2\pi n \left( \frac{n}{e} \right)^n})^2} \left( \frac{1}{2} \right)^{2n} = \frac{1}{\sqrt{\pi n}} \]

- Recall an important fact from calculus:

\[ \sum_{n\geq1} \frac{1}{n^a} = \begin{cases} \infty & a \leq 1 \\ \text{finite} & a > 1 \end{cases} \]

- Hence

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty} P (S_{2n} = 0) = \infty \]
Conclusion: 1-D random walk always comes back back

\[ P_0 = P \text{ (the drunkard never returns to 0)} = 0, \]
\[ P(E) = P \text{ (the drunkard returns to 0 finitely often)} = 0. \]

i.e.,

\[ P \text{ (the drunkard returns to 0 eventually)} = 1, \]
\[ P \text{ (the drunkard returns to 0 infinitely often)} = 1. \]
2 and 3-dim random walks
Drunkard’s Walks on the plane

A drunkard walks randomly in an idealized 2-dimensional city. The city is infinite and arranged in an equally-spaced square grid. At every intersection, the drunkard chooses one of the 4 directions: N/S/E/W, with equal probability. Formally, this is a random walk on $\mathbb{Z}^2$. 

![Diagram of a 2D grid with arrows indicating directions: N/S/E/W]
Drunkard’s Walks on the plane

A drunkard walks randomly in an idealized 2-dimensional city. The city is infinite and arranged in an equally-spaced square grid. At every intersection, the drunkard chooses one of the 4 directions: N/S/E/W, with equal probability. Formally, this is a random walk on $\mathbb{Z}^2$.

**Question**

What is the probability of eventual return?
Equivalent view
We can consider the walk which moves to NE/NW/SW/SW:
\[(x, y) \rightarrow (x \pm 1, y \pm 1)\]
Equivalent view

We can consider the walk which moves to NE/NW/SW/SW: $(x, y) \rightarrow (x \pm 1, y \pm 1)$
Equivalent view

We can consider the walk which moves to NE/NW/SW/SW:

$$(x, y) \rightarrow (x \pm 1, y \pm 1)$$
2 Dimensional Random Walk

• Two independent sequences of independent random variables:

\[ P (X_i = 1) = P (X_i = -1) = \frac{1}{2} \] horizontal steps

\[ P (Y_i = 1) = P (Y_i = -1) = \frac{1}{2} \] vertical steps

• Position at time \( n \):

\[ S_n = \sum_{i=1}^{n} X_i \] horizontal coordinate

\[ T_n = \sum_{i=1}^{n} Y_i \] vertical coordinate
Apply the same reasoning

- Let $P_0 = P(\text{never returns})$. 

It boils down to $P_n \geq \frac{39}{48}$.
Apply the same reasoning

- Let $P_0 = P(\text{never returns})$.
- Define the event:

$$E = \{\text{drunkard returns to origin finitely often}\}$$
Apply the same reasoning

- Let \( P_0 = P(\text{never returns}) \).
- Define the event:

\[
E = \{\text{drunkard returns to origin finitely often}\}
\]

- Key identity is the same as in one dimension:

\[
P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) \times P_0
\]

It boils down to \( \sum_{n \geq 0} P(S_{2n} = 0, T_{2n} = 0) \overset{?}{=} \infty \)
Using Stirling’s formula again

\[ P(S_{2n} = 0, T_{2n} = 0) = P(S_{2n} = 0) P(T_{2n} = 0) \sim \left( \frac{1}{\sqrt{\pi n}} \right)^2 \]

**Important facts:**

- \( P(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}} \)
- \( \sum_{n=1}^{\infty} \frac{1}{n} = +\infty! \)

If \( P_0 \neq 0 \), then \( P(E) = +\infty \). Contradiction! Thus \( P_0 = 0 \)
Conclusion: 2-D random walk always comes back

\[ P_0 = P(\text{the drunkard never returns}) = 0, \]
\[ P(E) = P(\text{the drunkard returns finitely often}) = 0. \]

i.e.,

\[ P(\text{the drunkard returns eventually}) = 1, \]
\[ P(\text{the drunkard returns infinitely often}) = 1. \]
Drunkard’s Walks in space

A drunkard walks randomly in an idealized 3-dimensional city. The city is infinite and arranged in an equally-spaced cubic grid. At every intersection, the drunkard chooses one of the 6 directions: up/down/left/right/back/forth, with equal probability. Formally, this is a random walk on $\mathbb{Z}^3$. 
(Simplified) 3-dimensional Walk

Three independent sequences of independent random variables:

\[
P(X_i = 1) = P(X_i = -1) = \frac{1}{2},
\]

\[
P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2},
\]

\[
P(Z_i = 1) = P(Z_i = -1) = \frac{1}{2}.
\]

Position at time \(n\):

\[
S_n = \sum_{i=1}^{n} X_i, \quad T_n = \sum_{i=1}^{n} Y_i, \quad U_n = \sum_{i=1}^{n} Z_i.
\]

8 directions: \((x, y, z) \rightarrow (x \pm 1, y \pm 1, z \pm 1)\)
Key Difference

- The same reasoning leads to

\[ P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) \cdot P_0, \]

but now this is finite!

since

\[ P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) = P(S_{2n} = 0) \cdot P(T_{2n} = 0) \cdot P(U_{2n} = 0) \]

\[ \sim \left( \frac{1}{\sqrt{\pi n}} \right)^3 \]

and \( \sum_{n=1}^{\infty} \frac{1}{n^{1.5}} \approx 2.6 < \infty \)
Key Difference

- The same reasoning leads to

\[ P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) \cdot P_0, \]

but now this is finite!

since

\[ P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) = P(S_{2n} = 0) \cdot P(T_{2n} = 0) \cdot P(U_{2n} = 0) \]

\[ \sim \left( \frac{1}{\sqrt{\pi n}} \right)^3 \]

and \[ \sum_{n=1}^{\infty} \frac{1}{n^{1.5}} \approx 2.6 < \infty \]

- Thus it is not immediately clear whether \( P_0 = 0 \) or not.
In fact: $P_0 > 0$
In fact: $P_0 > 0$

- Suppose, for the sake of contradiction, that $P_0 = 0$. 
In fact: $P_0 > 0$

- Suppose, for the sake of contradiction, that $P_0 = 0$.
- Then $P(E) = P(\text{return to } (0,0,0) \text{ finitely often}) = 0$, i.e.,

$$P(\text{return to } (0,0,0) \text{ infinitely often}) = 1$$
In fact: $P_0 > 0$

- Suppose, for the sake of contradiction, that $P_0 = 0$.
- Then $P(E) = P(\text{return to } (0,0,0) \text{ finitely often}) = 0$, i.e.,

$$P(\text{return to } (0,0,0) \text{ infinitely often}) = 1$$

- Let $X = \text{number of returns}$. Then

$$X = \sum_{n \geq 0} 1_{\{S_{2n}=0,T_{2n}=0,U_{2n}=0\}}$$
In fact: $P_0 > 0$

- Suppose, for the sake of contradiction, that $P_0 = 0$.
- Then $P(E) = P(\text{return to } (0,0,0) \text{ finitely often}) = 0$, i.e.,
  \[
P(\text{return to } (0,0,0) \text{ infinitely often}) = 1
  \]
- Let $X = \text{number of returns}$. Then
  \[
  X = \sum_{n \geq 0} 1_{\{S_{2n}=0,T_{2n}=0,U_{2n}=0\}}
  \]
- We know $X = +\infty$ with probability 1, but
  \[
  E(X) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) < +\infty
  \]
  contradiction!
Conclusion: 3-D random walk might not return

\[ P(\text{never return}) > 0^\dagger \]

\[^\dagger P(\text{never return}) \approx 72\% \]
Conclusion: 3-D random walk might not return

\[ P(\text{never return}) > 0^\dagger \]

The same holds for

- the original 3-D walk with 6 directions (Grinstead-Snell, Sec 12.1 Ex 14): \( P(\text{never return}) \approx 66\% \)
- walks in higher dimensions

\[ ^\dagger P(\text{never return}) \approx 72\% \]
Summary

What we have learned: a dichotomy

\[
\sum_{k \geq 0} P(\text{return at time } k) = \infty \iff P(\text{never return}) = 0
\]

\[
\sum_{k \geq 0} P(\text{return at time } k) < \infty \iff P(\text{never return}) > 0
\]

This is applicable to analyzing other walks, e.g., asymmetric ones (HW)
More precisely

- If \( \sum_{k \geq 0} P(\text{return at time } k) = \infty \), then

\[
P(\text{return finitely often}) = 0
\]

\[
P(\text{never return}) = 0
\]

- If \( \sum_{k \geq 0} P(\text{return at time } k) < \infty \), since \( EX < \infty \), \( X \) is finite with probability one, then

\[
P(\text{return finitely often}) = 1
\]

\[
P(\text{never return}) = \frac{1}{\sum_{k \geq 0} P(\text{return at time } k)}
\]