S&DS 241 Lecture 15

Functions of continuous random variables

B-H 8.1

Overview

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- Let $Y = g(X)$. How to find its PDF f_Y?

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- Linear: $Y = aX + b$.
- Nonlinear: $Y = g(X)$, e.g., $Y = X^2$.

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- Linear: $Y = aX + b$.
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Strategy (same as in last lecture):

1. Find CDF F_Y
2. Differentiate to get PDF $f_Y = F'_Y$

Example: linear function

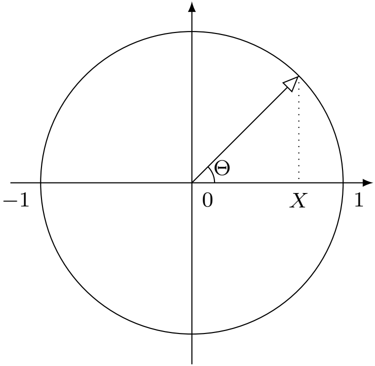
- X = temperature tomorrow in Fahrenheit
- Y = temperature tomorrow in Celcius

$$Y = \frac{5}{9}(X - 32)$$

Given f_X, how to find f_Y?

Example: nonlinear function

- Angle of a randomly spinned wheel



Then $\Theta \sim$ Unif$(0, 2\pi)$

- Horizontal position:

$$X = \cos(\Theta)$$

How to find f_X? Is it uniform?

Shifting and scaling

Shifting and scaling

Let $Y = aX + b$, where $a \neq 0$. Find CDF then PDF:

- Suppose $a > 0$.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) \\ &= F_X\left(\frac{y-b}{a}\right) \\ \implies f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \end{aligned}$$

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$$= 1 - F_X\left(\frac{y-b}{a}\right)$$
$$\implies f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{1}{a}f_X\left(\frac{y-b}{a}\right)$$

In general

Let $Y = aX + b$, where $a \neq 0$. Then

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Example

Let $X \sim \text{Unif}(0,1)$. Let $Y = aX + b$, where $a > 0$. Then

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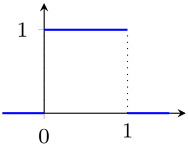

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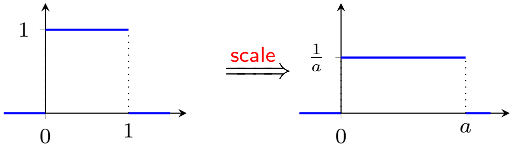

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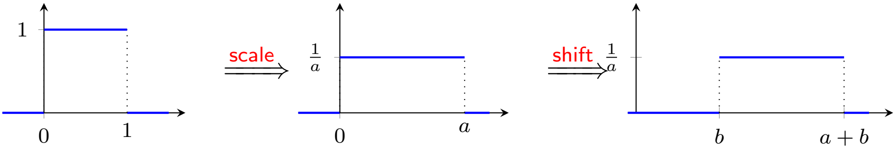

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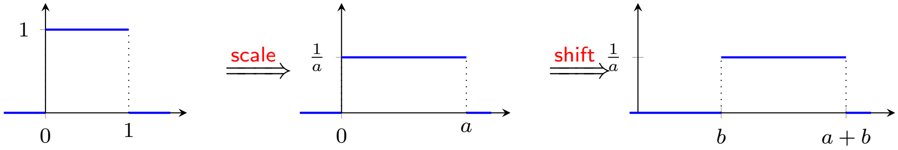

Hence $Y \sim \text{Unif}(b, a+b)$

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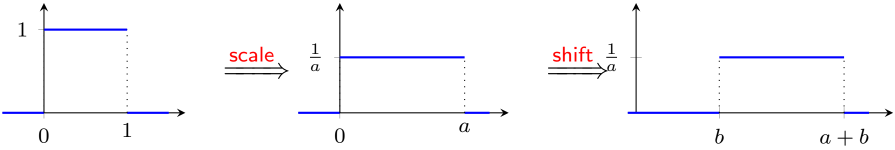


Hence $Y \sim \text{Unif}(b, a+b)$

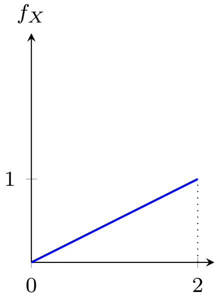
Summary

Step 1. Vertically "compress" by a factor of $1/a$;
horizontally "stretch" by a factor of a

Step 2. Horizontally shift by b

One more example

Let $Y = X/2 + 1$.



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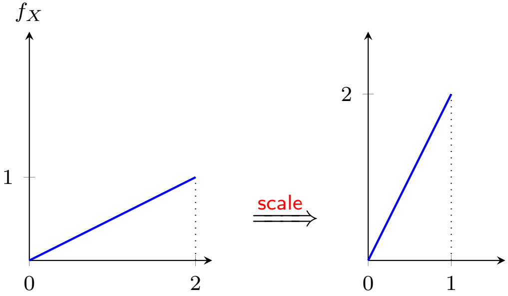

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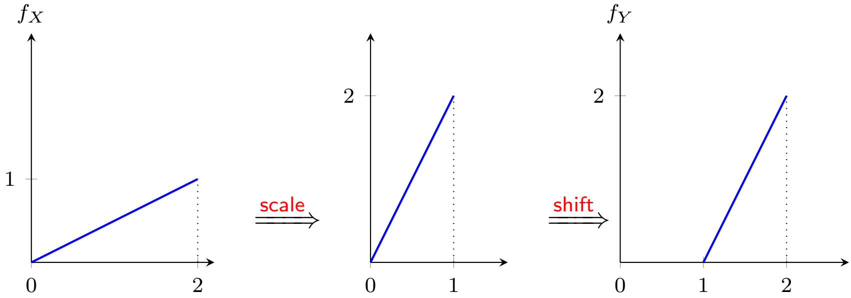

Standardization

Let $E(X) = \mu$ and $\mathrm{SD}(X) = \sigma$. Then

$$Y = \frac{X - \mu}{\sigma}$$

has zero mean and unit variance, called the standardized version of X.

Mean and variance

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Let $Y = aX + b$. Say $a > 0$,

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \frac{y}{a} f_X\left(\frac{y-b}{a}\right) dy \\ &\overset{x=(y-a)/b}{=} \int_{-\infty}^{\infty} (ax+b) f_X(x) dx \\ &= a \underbrace{\int_{-\infty}^{\infty} x f_X(x) dx}_{E(X)} + b. \end{aligned}$$

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Variance: exercise.

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$$f_Y(y) = \lambda f_X(\lambda y) = \begin{cases} \lambda e^{-\lambda y} & y > 0 \\ 0 & \text{else} \end{cases}$$

  So $Y \sim \text{Expo}(\lambda)$.
- The family of exponential distribution is related by scaling.

Nonlinear operations

Monotone transformation

Let $Y = g(X)$.

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- Suppose g is monotonically increasing:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) \\ &= F_X(g^{-1}(y)) \end{aligned}$$

$$\implies f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y))$$

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- Suppose g is monotonically decreasing:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(g(X) \leq y) = P(X \geq g^{-1}(y)) \\ &= 1 - F_X(g^{-1}(y)) \\ \implies f_Y(y) &= -\frac{d}{dy} F_X(g^{-1}(y)) = -\frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y)) \end{aligned}$$

Monotone transformation

Summary: Let $Y = g(X)$, where g is monotone. Then

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Special case: $Y = aX + b$, where $a \neq 0$. Then

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Example

Let $X \sim \text{Unif}(0, 1)$.

- Let $Y = -\ln(X)$, which takes positive values. For $y > 0$,

$$\begin{aligned} F_Y(y) &= P(-\ln(X) \leq y) = P(X \geq e^{-y}) = 1 - e^{-y} \\ \implies f_Y(y) &= F_Y'(y) = e^{-y}. \end{aligned}$$

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- Directly invoking formula yields the same

Common mistakes

- Since $Y = aX + b$, then $f_Y = af_X + b$.
- Since $Y = X^2$, then $f_Y = f_X^2$.

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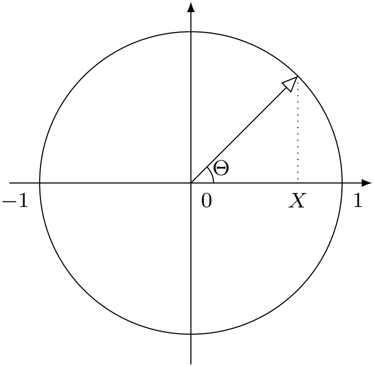
Final example

- Angle of a randomly spinned wheel: $\Theta \sim \text{Unif}(0, 2\pi)$



- Horizontal position:

$$X = \cos(\Theta)$$

How to find f_X? Is it uniform over $[-1, 1]$?

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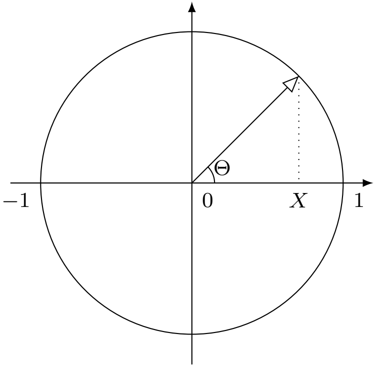


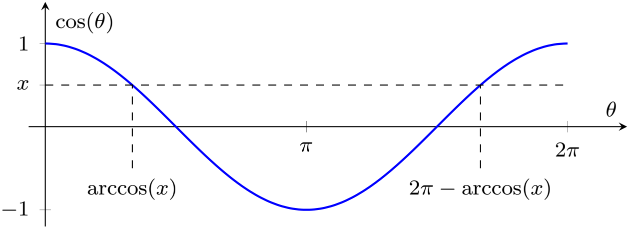
- Horizontal position:

$$X = \cos(\Theta)$$

How to find f_X? Is it uniform over $[-1, 1]$?

- $\cos$ is NOT monotone

Solution



Note that for $x \in [-1, 1]$,

$$\cos(\theta) \leq x \Leftrightarrow \arccos(x) \leq \theta \leq 2\pi - \arccos(x)$$

Thus

$$\begin{aligned} F_X(x) &= P(\cos(\Theta) \leq x) = P(\arccos(x) \leq \Theta \leq 2\pi - \arccos(x)) \\ &= \frac{2\pi - 2\arccos(x)}{2\pi} = 1 - \frac{1}{\pi}\arccos(x) \end{aligned}$$

PDF

$$\boxed{f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad -1 < x < 1}$$

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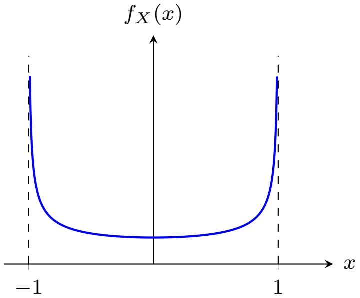

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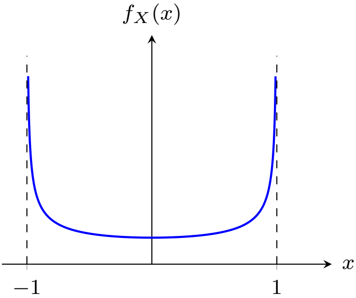

This is highly non-uniform! (PDF is unbounded, similarly to BH Chap 5 Prob 7 in HW)

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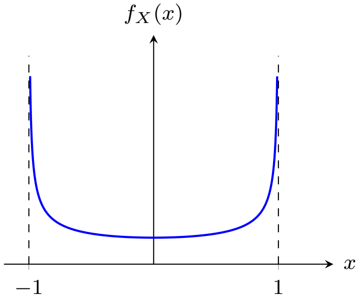

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- Why X is more likely to be near the end points than in the middle?
- Imagine the disk spins with constant angular velocity.
- The horizontal velocity $\propto \sin(\theta)$, which is zero at the end points and maximal in the middle. So it is passes over $x = 0$ quickly and "stays" near $x = \pm 1$.

Bonus

What is the PDF of the vertical coordinate:

$$Y = \sin(\Theta)$$

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Same as X, by symmetry!

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- If g is monotone, then

$$\boxed{f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))}$$

  Special case: $Y = aX + b$, where $a \neq 0$. Then

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- If g is not monotone
  - Find the CDF of Y by simplifying the event $\{g(X) \leq y\}$ in terms of the range of X
  - Take derivative to find PDF of Y