S&DS 241 Lecture 20

2-D transformation, polar coordinates, order statistics

B-H: 8.1,8.6

2-D Transformation

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a one-to-one and differentiable mapping. Let

$$\begin{pmatrix} U \\ V \end{pmatrix} = g \begin{pmatrix} X \\ Y \end{pmatrix}$$

How to find $f_{UV}(u, v)$ based on $f_{XY}(x, y)$?

Examples:

- Linear map: $U = X + Y$, $V = X - Y$
- Polar coordinates: $U = R$, $V = \Theta$

Linear transform and inverse

$$\begin{pmatrix} U \\ V \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an invertible matrix, or equivalently, the determinant is non-zero:

$$\det(A) = ad - bc \neq 0$$

Then

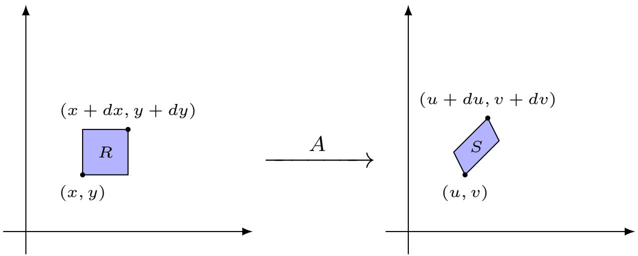
$$\begin{pmatrix} X \\ Y \end{pmatrix} = A^{-1} \begin{pmatrix} U \\ V \end{pmatrix}$$

where

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Strategy

Look at an infinitesimal region under this change of variables

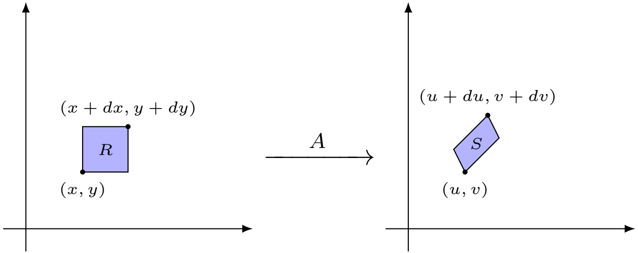


$$\underbrace{P((X,Y) \in R)}_{\approx f_{XY}(x,y)\cdot \text{area}(R)} = \underbrace{P((U,V) \in S)}_{\approx f_{UV}(u,v)\cdot \text{area}(S)}$$

So

$$f_{UV}(u,v) = f_{XY}(x,y)\frac{\text{area}(R)}{\text{area}(S)}$$

Joint PDF



Linear algebra:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = A \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \text{area}(S) = \text{area}(R) \cdot |\det(A)|$$

Therefore

$$\boxed{f_{UV}(u,v) = \frac{1}{|\det(A)|} f_{XY}(x,y)}$$

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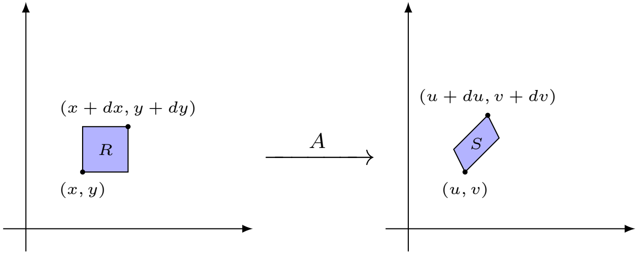

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Therefore

$$\boxed{f_{UV}(u,v) = \frac{1}{|\det(A)|} f_{XY}(A^{-1}(u,v))}$$

Example

$$\begin{cases} U = X + Y \\ V = X - Y \end{cases}, \text{ i.e. } \quad \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Then $\det(A) = -2$ and

$$f_{UV}(u, v) = \frac{1}{2} f_{XY}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

Example (continued)

$$\begin{cases} U = X + Y \\ V = X - Y \end{cases} \implies f_{UV}(u, v) = \frac{1}{2} f_{XY}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

Example

Suppose X, Y are iid $N(0, 1)$, so $f_{XY}(x, y) = \varphi(x)\varphi(y) = \frac{1}{2\pi} \exp(-\frac{x^2+y^2}{2})$.

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Then

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{4\pi}\exp\left\{-\frac{1}{2}\left(\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right)\right\} \\ &= \frac{1}{4\pi}\exp\left(-\frac{u^2+v^2}{4}\right) = \underbrace{\frac{1}{\sqrt{4\pi}}\exp\left(-\frac{u^2}{4}\right)}_{f_U(u)} \underbrace{\frac{1}{\sqrt{4\pi}}\exp\left(-\frac{v^2}{4}\right)}_{f_V(v)} \end{aligned}$$

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i.e. U, V are iid $N(0,2)$ (cf. Lec19, where we verified the marginals)

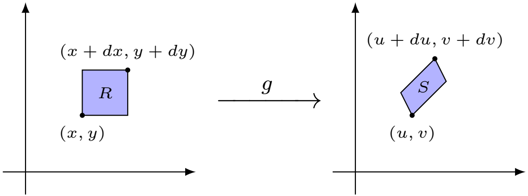
Linear transformation of normals

More generally (B-H §7.5):

- Orthogonal transformation of independent normal RVs remains independent normal

Non-linear transform: $(U, V) = g(X, Y)$

Look at an infinitestimal region under change of variables



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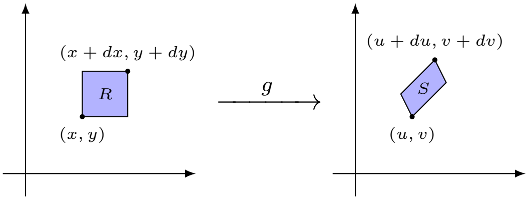


$$\begin{pmatrix} du \\ dv \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}}_{\frac{\partial(u,v)}{\partial(x,y)}:\ \text{Jacobian}} \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \text{area}(S) = \text{area}(R) \left| \det \frac{\partial(u,v)}{\partial(x,y)} \right|$$

Therefore

$$\boxed{f_{UV}(u,v) = \frac{f_{XY}(g^{-1}(u,v))}{\left|\det \frac{\partial(u,v)}{\partial(x,y)}\right|} = \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| f_{XY}(g^{-1}(u,v))}$$

since the Jacobian matrix $\frac{\partial(u,v)}{\partial(x,y)}$ and $\frac{\partial(x,y)}{\partial(u,v)}$ are inverse to each other.

Polar coordinates

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases}$$

where $X, Y \in \mathbb{R}$ and $R \in [0, \infty)$ and $\Theta \in [0, 2\pi)$.

Jacobian: $$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(r\cos\theta, r\sin\theta)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and

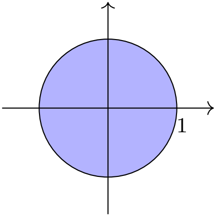
$$\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Thus

$$\boxed{f_{R\Theta}(r,\theta) = r f_{XY}(r\cos\theta, r\sin\theta)}$$

(this also follows from $dxdy = rdrd\theta$)

Example: Dart



- Let (X, Y) denote the coordinate of the dart, which is uniformly distributed over the board (a disk of unit radius). Then

$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{else} \end{cases}$$

- Let (R, Θ) denote the polar coordinate of the dart. Then

$$f_{R\Theta}(r,\theta) = r f_{XY}(r\cos\theta, r\sin\theta) = \begin{cases} \frac{r}{\pi} & 0 < r < 1, 0 < \theta < 2\pi \\ 0 & \text{else} \end{cases}$$

Example: Dart

- Joint PDFs of radius R and angle Θ:

$$f_{R\Theta}(r,\theta) = r f_{XY}(r\cos\theta, r\sin\theta) = \begin{cases} \frac{r}{\pi} & 0 < r < 1, 0 < \theta < 2\pi \\ 0 & \text{else} \end{cases}$$

- Marginal PDFs of radius and angle:

$$f_R(r) = \begin{cases} 2r & 0 < r < 1 \\ 0 & \text{else} \end{cases} \qquad f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 < \theta < 2\pi \\ 0 & \text{else} \end{cases}$$

- Observations:
  - R and Θ and independent, since $f_{R\Theta}(r,\theta) = f_R(r) f_\Theta(\theta)$
  - The angle Θ is uniform
  - The radius R is non-uniform (more likely to be on the rim)

Example: indepedent normals

- Let (X, Y) be iid $N(0,1)$. Then

$$f_{XY}(x,y) = \varphi(x)\varphi(y) = \frac{1}{2\pi} \exp\Big(- \frac{x^2+y^2}{2} \Big)$$

- Let (R, Θ) denote the polar coordinates. Then

$$f_{R\Theta}(r,\theta) = r f_{XY}(r\cos\theta, r\sin\theta) = \underbrace{\frac{1}{2\pi}}_{f_\Theta(\theta)} \underbrace{r \exp\Big(- \frac{r^2}{2} \Big)}_{f_R(r)}$$

- R and Θ are independent:
  - $\Theta \sim \text{Unif}(0, 2\pi)$
  - $R = \sqrt{X^2+Y^2}$ has PDF $re^{-r^2/2}$ (Rayleigh distribution; B-H Example 5.1.7)
  - $R^2 = X^2 + Y^2$ has an exponential distribution, Expo$(1/2)$

Application

- How to generate two independent standard normal random variables?

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- Quantile transform (Lec 14): Let U, V be iid Unif$(0,1)$. Then

$$\begin{cases} X = \Phi^{-1}(U) \\ Y = \Phi^{-1}(V) \end{cases}$$

are iid $N(0,1)$, where Φ stands for standard normal CDF

Box-Muller method (B-H Example 8.1.9)

- Using polar coordinates:

$$\begin{cases} X = \sqrt{-2\ln U}\cos(2\pi V) \\ Y = \sqrt{-2\ln U}\sin(2\pi V) \end{cases}$$

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- Why this works?
  - Radius:

$$U \sim \text{Unif}(0,1) \Rightarrow -\ln U \sim \text{Expo}(1) \Rightarrow -2\ln U \sim \text{Expo}(1/2)$$
$$\Rightarrow \sqrt{-2\ln U} \sim \text{Rayleigh}$$

  - Angle: $V \sim \text{Unif}(0,1) \Rightarrow 2\pi V \sim \text{Unif}(0,2\pi)$

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  - Angle: $V \sim \text{Unif}(0,1) \Rightarrow 2\pi V \sim \text{Unif}(0, 2\pi)$
- What is the potential advantage of Box-Muller over quantile transform? No need to evaulate Φ^{-1} which has no close-form formula

What happens if g is not one-to-one?

Example: order statistics

Given X and Y, let

$$\begin{cases} U = \min\{X, Y\} \\ V = \max\{X, Y\} \end{cases}$$

How to express $f_{UV}(u, v)$ in terms of $f_{XY}(x, y)$?

- Clearly this is not a one-to-one mapping (in fact, two-to-one).
- U and V are dependent, since $U \leq V$ by definition.
- Support of (U, V): $S = \{(u, v) : u \leq v\}$

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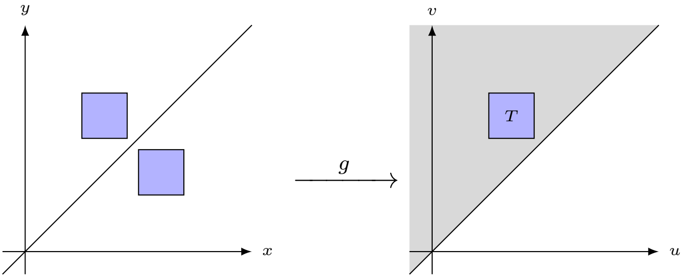
To find joint PDF, for any $T \subset S$, suppose we can express $P((U, V) \in T)$ as an integral:

$$P((U, V) \in T) = \iint_T f(u, v) du dv$$

then the integrand $f(u, v)$ must be the joint PDF $f_{UV}(u, v)$

Example: order statistics



$$
\begin{aligned}
P((U,V) \in T) &= P((X,Y) \in T) + P((Y,X) \in T) \\
&= \iint_T f_{XY}(u,v)\,dudv + \iint_T f_{XY}(v,u)\,dudv \\
&= \iint_T \underbrace{f_{XY}(u,v) + f_{XY}(v,u)}_{f_{UV}(u,v)}\,dudv
\end{aligned}
$$

Example: order statistics

- Joint PDF of order statistics of X, Y:

$$f_{UV}(u,v) = \begin{cases} f_{XY}(u,v) + f_{XY}(v,u) & u \leq v \\ 0 & \text{else} \end{cases}$$

- Example: X, Y are iid Unif$(0,1)$. Then

$$f_{UV}(u,v) = \begin{cases} 2 & 0 \leq u \leq v \leq 1 \\ 0 & \text{else} \end{cases}$$

- Order statistics of multiple RVs: B-H §8.6