S&DS 241 Lecture 21

Covariance and correlation

B-H: 7.3, 10.1

Given a random variable X

Recall mean and variance

- EX: mean value
- $\mathrm{Var}(X) = E(X - EX)^2$: mean-square deviation around the mean

Given two random variables X and Y

Suppose

- X has mean EX and $\text{Var}(X) = \sigma_X^2$
- Y has mean EY and $\text{Var}(Y) = \sigma_Y^2$

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Two important quantities:

- Covariance:

$$\mathrm{Cov}(X, Y) = E((X - EX)(Y - EY))$$

  summary statistic for the "tendency" of X and Y to move in the same direction

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- Correlation coefficient:

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

  (Denoted by Corr(X, Y) in B-H)

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  - e.g. X=yield of crop, Y=market price
  - e.g. roll a die for 100 times, X=number of 1's, Y=number of 6's

Uncorrelated

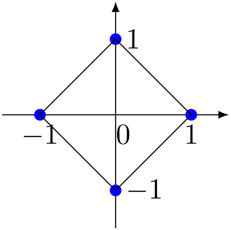
We say X and Y are <u>uncorrelated</u> if

$$\begin{aligned} &\mathrm{Cov}(X, Y) = 0 \\ \Leftrightarrow\ &\rho(X, Y) = 0 \\ \Leftrightarrow\ &E(XY) = (EX)(EY) \end{aligned}$$

Independent versus uncorrelated

- Independent $\Rightarrow$ uncorrelated: $E(XY) = (EX)(EY)$.
- Uncorrelated $\not\Rightarrow$ independent:

  Example (HW3): Choose a point uniformly at random to be one of the four vertices of the diamond below. Let (X, Y) denote its coordinate.



  Then $E(XY) = 0 = EX = EY$, but X and Y are dependent

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$$\mathrm{Cov}(aX+b, cY+d) = ac\mathrm{Cov}(X, Y)$$

and

$$\rho(aX+b, cY+d) = \rho(X, Y), \quad \text{provided that } a, c > 0$$

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3. Bilinearity:

$$\begin{aligned} &\mathrm{Cov}(X+Y, W+Z) \\ =\ &\mathrm{Cov}(X, W) + \mathrm{Cov}(Y, W) + \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z) \end{aligned}$$

Properties

4. $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$, or equivalently

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho(X,Y)\sigma_X\sigma_Y$$

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5. $$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$$

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6. $$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

Variance of a sum

Corollary: When do variances add up?

- Suppose $X_1, \ldots, X_n$ are uncorrelated, that is, $\mathrm{Cov}(X_i, X_j) = 0$ whenever $i \neq j$. Then

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$$

Special case: $X_1, \ldots, X_n$ are independent

Example

Let X be a random variable (signal) with unit variance. Let Y be a noisy observation of X

$$Y = X + Z$$

where the noise Z is independent of X and has variance σ^2.

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- Covariance

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, X+Z) = \underbrace{\mathrm{Cov}(X,X)}_{\mathrm{Var}(X)} + \underbrace{\mathrm{Cov}(X,Z)}_{0} = 1$$

- Correlation coefficient

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{1}{\sqrt{1+\sigma^2}}$$

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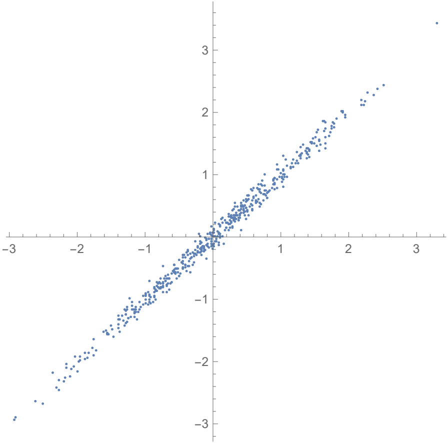
$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{1}{\sqrt{1 + \sigma^2}}$$

- X and Y are highly (positively) correlated when σ is small (less noisy)
- X and Y are almost uncorrelated when σ is large (very noisy)

Example

Scatter plot of 500 independent samples of (X, Y):



$$\sigma = 0.1, \rho = 0.995$$

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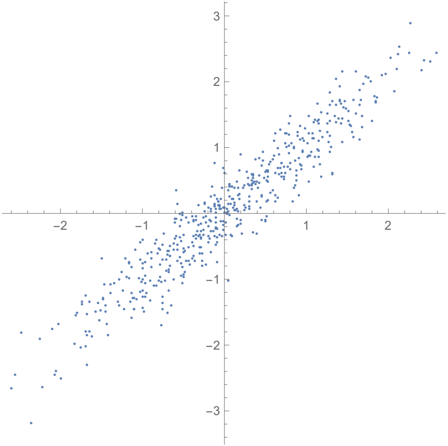

$$\sigma = 0.3, \rho = 0.96$$

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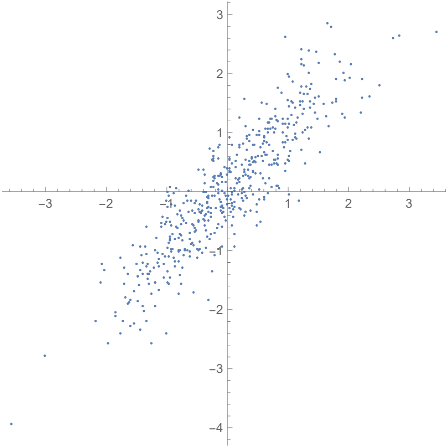

$$\sigma = 0.5, \rho = 0.89$$

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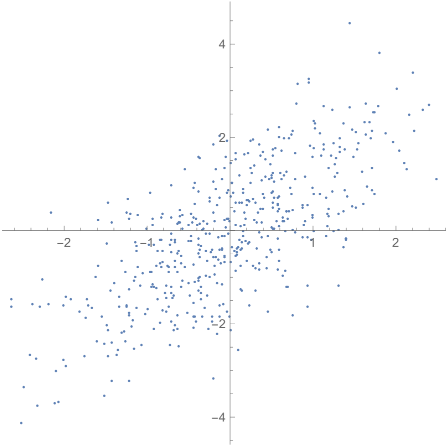

$$\sigma = 1, \rho = 0.71$$

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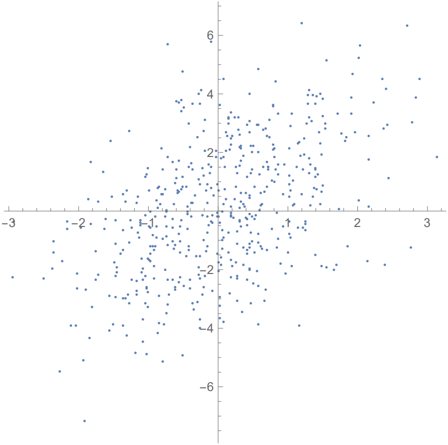

$$\sigma = 2, \rho = 0.45$$

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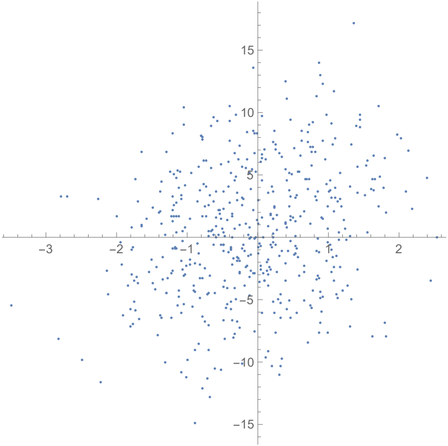


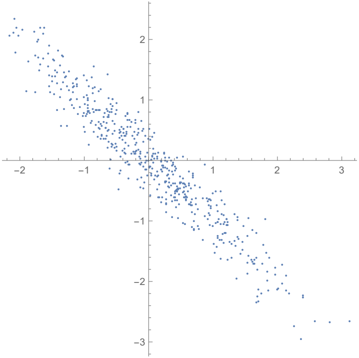
$$\sigma = 5, \rho = 0.2$$

Example

Similarly, if $Y = -X + Z$, then

$$\rho(X, Y) = -\frac{1}{\sqrt{1 + \sigma^2}}$$

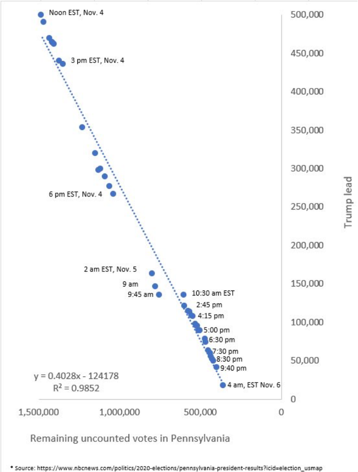
Scatter plot of 500 independent samples of (X, Y):



$$\sigma = 0.3, \rho = -0.96$$

Intuition

- If X and Y are highly correlated (i.e., $\rho(X, Y) \approx \pm 1$), then they are approximately related using a straight line. Thus we can predict one by the other by a linear equation. This is the idea of linear regression (in its most basic form)



Caveat

- Correlation only capture "linear" dependence.
  - Example: $X \sim \text{Unif}(-1,1)$, $Y = X^2$. $\text{Cov}(X,Y) = 0$ (Exercise)
  - Cannot predict Y using linear instrument, but X completely determines Y (non-linearly)

Example: exam score

Assume X (Pset score), Y (midterm score) and Z (final score) are independent and Unif$(0, 100)$. Total score:

$$S = 0.3X + 0.3Y + 0.4Z$$

Then

- $\mathrm{Var}(X) = \mathrm{Var}(Y) = \mathrm{Var}(Z) = \sigma^2$ and $\mathrm{Var}(S) = (0.3^2 + 0.3^2 + 0.4^2)\sigma^2$
- Correlation between Pset and total

$$\rho(X, S) = \frac{\mathrm{Cov}(X, 0.3X + 0.3Y + 0.4Z)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(S)}} = \frac{0.3}{\sqrt{0.3^2 + 0.3^2 + 0.4^2}}$$
$$\approx 0.51$$

- Similarly, $\rho(Y, S) \approx 0.51$, $\rho(Z, S) \approx 0.69$.

Example: dice

Roll a fair die for n times, X=number of ⚀, Y=number of ⚅. Find $\rho(X, Y)$

- $X \sim \text{Bin}(n, 1/6)$, $Y \sim \text{Bin}(n, 1/6)$

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- $X \sim \text{Bin}(n, 1/6)$, $Y \sim \text{Bin}(n, 1/6)$
- Are they independent? No.
- Decompose binomial as sum of independent Bernoullis:

$$X = \sum_{i=1}^{n} X_i, \quad X_i = \begin{cases} 1 & i\text{th toss is 1} \\ 0 & \text{else} \end{cases}$$

$$Y = \sum_{i=1}^{n} Y_i, \quad Y_i = \begin{cases} 1 & i\text{th toss is 6} \\ 0 & \text{else} \end{cases}$$

- Each X_i and Y_i are dependent; for $i \neq j$, X_i and Y_j are independent

Example: dice

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(X_1 + \ldots + X_n, Y_1 + \ldots + Y_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, Y_j)$$

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$$= \sum_{i=1}^{n} \operatorname{Cov}(X_i, Y_i) + \sum_{i \neq j} \underbrace{\operatorname{Cov}(X_i, Y_j)}_{0, \text{ by independence}}$$

Example: dice

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathrm{Cov}(X_1+\ldots+X_n, Y_1+\ldots+Y_n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}(X_i,Y_j)\\
&= \sum_{i=1}^{n}\mathrm{Cov}(X_i,Y_i) + \sum_{i\neq j}\underbrace{\mathrm{Cov}(X_i,Y_j)}_{0,\text{ by independence}}\\
&= \sum_{i=1}^{n}\underbrace{E(X_iY_i)-E(X_i)E(Y_i)}_{0-\frac{1}{6}\times\frac{1}{6}} = -\frac{n}{36}
\end{aligned}$$

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$$= \sum_{i=1}^{n}\underbrace{E(X_iY_i) - E(X_i)E(Y_i)}_{0-\frac{1}{6}\times\frac{1}{6}} = -\frac{n}{36}$$

and $\mathrm{Var}(X) = \mathrm{Var}(Y) = n \times \frac{1}{6} \times \frac{5}{6}$. Thus

$$\rho(X,Y) = \frac{-\frac{n}{36}}{n\frac{1}{6}\frac{5}{6}} = -\frac{1}{5}$$

Why $\rho \in [-1, 1]$?

Cauchy-Schwarz inequality (B-H 10.1)

Theorem

For any random variables U, V*:*

$$(E(UV))^2 \leq E(U^2)E(V^2),$$

with equality if and only if $U = cV$ *for some constant* c*.*

Cauchy-Schwarz inequality (B-H 10.1)

Theorem

For any random variables U, V:

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with equality if and only if $U = cV$ for some constant c.

Corollary

For any random variables X, Y:

$$|\mathrm{Cov}(X,Y)| \leq \sigma_X \sigma_Y, \quad \textit{i.e.}\ |\rho(X,Y)| \leq 1$$

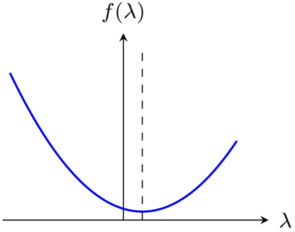
Furthermore,

- $\rho(X,Y) = 1 \Leftrightarrow Y = aX + b$ *for some* $a > 0$ *(perferctly positively correlated) : e.g.* X *is temperature in* $^\circ$*C and* Y *in* $^\circ$*F*
- $\rho(X,Y) = -1 \Leftrightarrow Y = aX + b$ *for some* $a < 0$ *(perferctly negatively correlated): e.g.* X*=number of Heads,* Y *= number of Tails*

Cauchy-Schwarz inequality: proof (optional)

Consider the function:

$$f(\lambda) = E(U - \lambda V)^2$$
$$= E(U^2) + \lambda^2 E(V^2) - 2\lambda E(UV)$$



- By definition, $f(\lambda) \geq 0$ for all λ. The minimum of the parabola is achieved at $\lambda_0 = \frac{E(UV)}{E(V^2)}$ to be

$$E(U^2) - \frac{(E(UV))^2}{E(V^2)},$$

  which must be non-negative.
- (The case of equality) Suppose the minimum is zero. Then $f(\lambda_0) = E(U - \lambda_0 V)^2 = 0$, which means that $U - \lambda_0 V$ is always zero, i.e., $U = \lambda_0 V$.

Application: portfolio optimization

- Two assets:
  - Stock return S: $\mu_S = 10\%$, $\sigma_S = 10\%$
  - Bond return B: $\mu_B = 5\%$, $\sigma_B = 5\%$
  - Correlation coefficient: $\rho(S, B) = -0.5$
  - Incentive to invest in bond: hedge the risk of stock!

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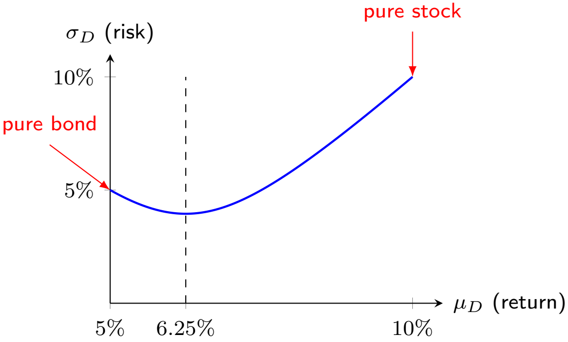
- Portfolio: invest λ fraction of funds in stock and $1 - \lambda$ in bond.
- Return: $D = \lambda S + (1 - \lambda) B$
  - Expected return:

$$\mu_D = \lambda \mu_S + (1 - \lambda) \mu_B$$

  - Variance:

$$\sigma_D^2 = \lambda^2 \sigma_S^2 + (1 - \lambda)^2 \sigma_B^2 + 2\lambda(1 - \lambda)\rho(S, B)\sigma_S \sigma_B$$

Performance of portfolio



- Minimal risk ($\sigma_D = 4\%$) occurred at $\mu_D = 6.25\%$: $\frac{1}{4}S + \frac{3}{4}B$
- As $\lambda \uparrow$, portfolio shifts to stock
  - return $\uparrow$; volatility first $\downarrow$ then $\uparrow$

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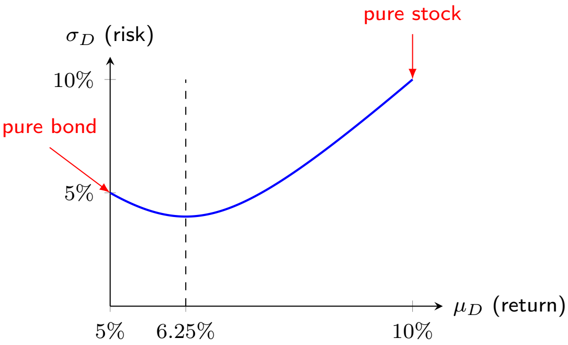

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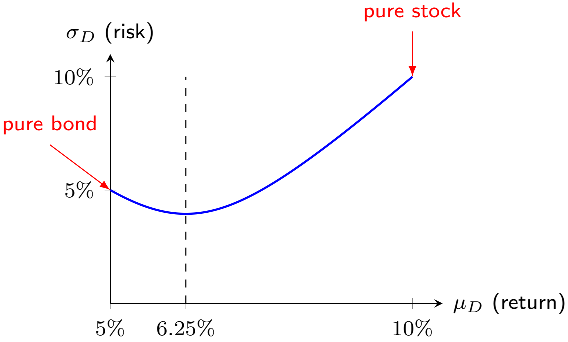
- A reasonable investor would not operate on the left of the minimum

Performance of portfolio



- Minimal risk ($\sigma_D = 4\%$) occurred at $\mu_D = 6.25\%$: $\frac{1}{4}S + \frac{3}{4}B$
- As $\lambda \uparrow$, portfolio shifts to stock
  - return $\uparrow$; volatility first $\downarrow$ then $\uparrow$
- A reasonable investor would not operate on the left of the minimum
- Things become more complicated with multiple assets (Markowitz portfolio optimization)

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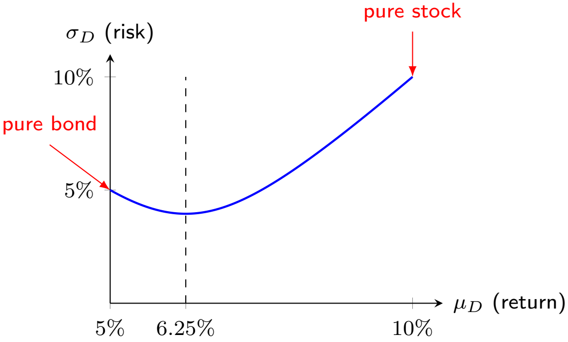

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- Can you go make a fortune after class? What's the catch?