Spring 2017
STAT 364 - Information Theory
Homework 2
Due: Feb 9, 2017
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1 Exercises

1. Suppose $Z_1, \dots Z_n$ are independent Poisson random variables with mean λ. Show that $\sum_{i=1}^n Z_i$ is a sufficient statistic of $(Z_1, \dots Z_n)$ for λ.

2. (Divergence of order statistics) Given $x^n = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $x_{(1)} \leq \dots \leq x_{(n)}$ denote the ordered entries. Let P, Q be distributions on $\mathbb{R}$ and $P_{X^n} = P^n, Q_{X^n} = Q^n$.

   (a) Prove that
   $$D(P_{X_{(1)},\dots,X_{(n)}} \| Q_{X_{(1)},\dots,X_{(n)}}) = nD(P\|Q). \tag{1}$$

   (b) Show that
   $$D(\text{Binom}(n,p)\|\text{Binom}(n,q)) = nd(p\|q).$$

   Hint: think about sufficient statistics.

3. Let (X, Y) be uniformly distributed in the unit ℓ_p-ball $B_p \triangleq \{(x,y) : |x|^p + |y|^p \leq 1\}$, where $p \in (0, \infty)$. Also define the ℓ_∞-ball $B_\infty \triangleq \{(x,y) : |x| \leq 1, |y| \leq 1\}$.

   (a) Compute $I(X;Y)$ for $p = 1/2$, $p = 1$ and $p = \infty$.

   (b) (Bonus) What do you think $I(X;Y)$ converges to as $p \to 0$. Can you prove it?

4. (Maximum entropy.) Prove that for any X taking values on $\mathbb{N} = \{1, 2, \dots\}$ such that $\mathbb{E}[X] < \infty$,
   $$H(X) \leq \mathbb{E}[X] h\left(\frac{1}{\mathbb{E}[X]}\right),$$
   maximized uniquely by the geometric distribution. Here as usual $h(\cdot)$ denotes the binary entropy function. *Hint*: Find an appropriate Q such that RHS - LHS = $D(P_X\|Q)$.

5. (Finiteness of entropy) We have shown that any $\mathbb{N}$-valued random variable X, with $\mathbb{E}[X] < \infty$ has $H(X) \leq \mathbb{E}[X]h(1/\mathbb{E}[X]) < \infty$. Next let us improve this result.

   (a) Show that $\mathbb{E}[\log X] < \infty \Rightarrow H(X) < \infty$.
   Moreover, show that the condition of X being integer-valued is not superfluous by giving a counterexample.

   (b) Show that if $k \mapsto P_X(k)$ is a decreasing sequence, then $H(X) < \infty \Rightarrow \mathbb{E}[\log X] < \infty$.
   Moreover, show that the monotonicity of pmf is not superfluous by giving a counterexample.

6. For any Gaussian random variable $X_{\mathtt{G}}$ and any random variable Y with finite variance, show that
   $$I(X_{\mathtt{G}}; Y_{\mathtt{G}}) \leq I(X_{\mathtt{G}}; Y)$$
   where $Y_{\mathtt{G}}$ is jointly Gaussian with $X_{\mathtt{G}}$ with the same mean and variance as Y and $E[X_{\mathtt{G}}Y_{\mathtt{G}}] = E[X_{\mathtt{G}}Y]$. Does the claim also hold if $Y_{\mathtt{G}}$ is Gaussian but not jointly Gaussian with $X_{\mathtt{G}}$?

2 Optional reading

1. Read [1, Chapter 2]

References

[1] T. Cover and J. Thomas, *Elements of Information Theory,* Second Edition, Wiley, 2006