Spring 2017

STAT 364 - Information Theory

Homework 6

Due: Apr 11, 2017 Tuesday in class

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1 Exercises

1. Find the capacity of the erasure-error channel (Figure 1 on page 1) with channel matrix

$$W = \begin{bmatrix} 1-2\delta & \delta & \delta \\ \delta & \delta & 1-2\delta \end{bmatrix}$$

where $0 \le \delta \le 1/2$.

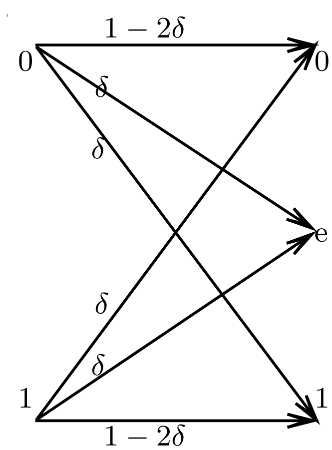

Figure 1: Binary erasure-error channel.

2. (Z-channel) Consider a discrete memoryless channel (DMC) with input and output alphabets $\mathcal{A} = \mathcal{B} = \{0, 1\}$ and

$$P_{Y|X} = \begin{pmatrix} 1 & 0 \\ \delta & 1-\delta \end{pmatrix}$$

Find the limit of the capacity **and** of the capacity achieving distributions P_X when $\delta \to 1$.

3. In the lectures we have shown that for additive noise, random linear codes achieves the same performance as Shannon's ensemble. The total number of possible generator matrices is 2^{nk}, which is significant smaller than double exponential, but still quite large. Now we show that without degrading the performance, we can reduce the number to 2^n by restricting to Toeplitz generator matrix G, i.e., $G_{ij} = G_{i-1,j-1}$ for all $i, j > 1$.

Consider the binary symmetry channel of blocklength n:

$$Y^n = X^n \oplus Z^n$$

where $Z^n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\delta)$.

(a) For any $1 \le k \le n$, show that there exists a linear code $f : \mathbb{F}_2^k \to \mathbb{F}_2^n$ with *Toeplitz* generator matrix, such that the probability of error is at most

$$P_e \le \mathbb{P}\left[\log_2 \frac{1}{P_{Z^n}(Z^n)} \ge n - k - \tau\right] + \exp(-\tau)$$

(b) Show that whenever $R = k/n$ is strictly below the capacity $C = 1 - h(\delta)$, the above bound on the probability of error vanishes as $n \to \infty$.

(c) How many $n \times k$ Toeplitz generator matrices are there? Compare with the number of $n \times k$ generator matrices.

Hint: Analogous to the proof of usual proof, first consider the usual random linear code ensemble plus random dithering, then argue that dithering can be removed without changing the performance of the codes. Show that codewords are pairwise independent and uniform.

4. (Sum of channels) Let W_1 and W_2 denote the channel matrices of discrete memoryless channel (DMC) $P_{Y_1|X_1}$ and $P_{Y_2|X_2}$ with capacity C_1 and C_2, respectively. The sum of the two channels is another DMC with channel matrix $\left[\begin{smallmatrix} W_1 & 0 \\ 0 & W_2 \end{smallmatrix}\right]$. Show that the capacity of the sum channel is given by

$$C = \log(\exp(C_1) + \exp(C_2)).$$

5. (Product of channels) The product channel of DMCs $P_{Y_1|X_1}$ and $P_{Y_2|X_2}$ is another DMC with random transformation $P_{Y_1Y_2|X_1X_2}$. Show that the capacity of the sum channel is given by

$$C = C_1 + C_2.$$

6. Consider a stationary memoryless additive <u>non-Gaussian</u> noise channel:

$$Y_i = X_i + Z_i\,, \qquad \mathbb{E}[Z_i] = 0, \quad \mathrm{Var}[Z_i] = 1$$

with the input constraint

$$\|x^n\|_2 \le \sqrt{nP} \qquad \Longleftrightarrow \qquad \sum_{i=1}^{n} x_i^2 \le nP\,.$$

(a) Prove that capacity $C(P)$ of this channel satisfies

$$\frac{1}{2}\log(1+P) \le C(P) \le \frac{1}{2}\log(1+P) + D(P_Z \| \mathcal{N}(0,1))\,,$$

where P_Z is the distribution of the noise. (Hints: Gaussian saddle point and the golden formula $I(X;Y) \le D(P_{Y|X} \| Q_Y | P_X)$.)

(b) If $D(P_Z \| \mathcal{N}(0,1)) = \infty$ (Z is very non-Gaussian), then it is possible that the capacity is infinite. Consider Z is ± 1 equiprobably. Show that the capacity is infinite by a) proving the maximal mutual information is infinite; b) giving an explicit scheme to achieve infinite capacity.

2 Optional reading

1. Read [1, Chapters 8-9]

References

[1] T. Cover and J. Thomas, *Elements of Information Theory,* Second Edition, Wiley, 2006