Spring 2017

STAT 364 - Information Theory

Homework 7

Due: Apr 20, 2017

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1. (Uniform-noise channel with peak power constraint) Let $A > 0$. Let X be independent of N, where N is uniformly distributed on $[-1, 1]$ and $|X| \le A$ almost surely.

   (a) Assume that A is a positive integer. Find the capacity

   $$\max_{X:|X|\le A \text{ a.s.}} I(X; X+N)$$

   and the maximizer.

   (b) (Bonus) What can you say about $A > 0$ in general?

2. Consider the *polygon channel* we discussed in the lecture, where the input and output alphabet are both $\{1, \ldots, L\}$, and $P_{Y|X}(b|a) > 0$ if and only if $b = a$ or $b = (a \mod L) + 1$. The confusability graph is a *cycle* of L vertices. *Rigorously* prove the following:

   (a) For all L, The zero-error capacity with feedback is $C_{fb,0} = \log \frac{L}{2}$.

   (b) For even L, the zero-error capacity without feedback $C_0 = \log \frac{L}{2}$.

   (c) Now consider the following channel, where the input and output alphabet are both $\{1, \ldots, L\}$, and $P_{Y|X}(b|a) > 0$ if and only if $b = a$ or $b = a + 1$. In this case the confusability graph is a *path* of L vertices. Show that the zero-error capacity is given by

   $$C_0 = \log \left\lceil \frac{L}{2} \right\rceil$$

   What is $C_{fb,0}$?

3. (BEC with feedback) Consider the stationary memoryless binary erasure channel with erasure probability δ and noiseless feedback. Design a *fixed*-blocklength[1] coding scheme achieving the capacity, i.e., find a scheme that sends k bits over n channel uses with noiseless feedback, such that the rate $\frac{k}{n}$ approaches the capacity $1-\delta$ when $n \to \infty$ and the maximal probability of error vanishes. Describe the encoding and decoding operations and *rigorously* prove your result.

   (Hint: Try retransmitting each bit until received.)

4. Consider a source $S^n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\frac{1}{2})$. Answer the following question as a function of n *when n is large.*

   (a) Design your best compressor which achieves one-bit distortion, i.e., $\mathbb{E}\|\hat{S}^n - S^n\|_{\text{Hamming}} \le 1$.[2] How many bits do you need? Is it optimal?

   (b) Design your best one-bit compressor. What is the smallest distortion you can get? Is it optimal?

[1] Variable blocklength codes are not allowed.

[2] The Hamming distance between two binary strings is the number of bits they differ in.