S&DS 602: Homework 2

Due Wednesday, September 11 at 2PM, via Gradescope

1. (Bennett's inequality) Suppose $\mathbb{E}X = 0$, $\operatorname{Var} X = \sigma^2$, and $X \leq b$ a.s. Show that

$$\log \mathbb{E} e^{\lambda X} \leq \frac{\sigma^2(e^{\lambda b} - 1 - \lambda b)}{b^2} \text{ for all } \lambda \geq 0.$$

[Hint: Show first that $u \mapsto (e^u - 1 - u)/u^2$ is nondecreasing.]
Letting $X_1, \ldots, X_n$ be i.i.d. and equal in law to X, show then that

$$\mathbb{P}\left[\sum_{i=1}^n X_i \geq t\right] \leq \exp\left(-\frac{n\sigma^2}{b^2} h\left(\frac{bt}{n\sigma^2}\right)\right) \text{ for all } t \geq 0$$

where $h(x) = (1+x)\log(1+x) - x$.

2. Let $Z_i \sim \text{Bernoulli}(p)$ where $p \leq 1/2$. For $t \in (0, 1-p)$, compare the lower bounds for

$$-\log \mathbb{P}\left[\frac{1}{n}\sum_{i=1}^n Z_i \geq t + p\right]$$

given by Hoeffding's inequality, Bernstein's inequality, and Bennett's inequality of Problem 1. As $n \to \infty$, for which $(p, t) = (p(n), t(n))$ does Bernstein's lower bound for $-\log \mathbb{P}[\frac{1}{n}\sum_{i=1}^n Z_i \geq t + p]$ improve over Hoeffding's by more than a constant factor? For which (p, t) does Bennett's improve over Bernstein's by more than a constant factor?

3. (Wainwright 2.18) Let $\psi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function that satisfies $\psi(0) = 0$. The ψ-Orlicz norm of a random variable X is defined as

$$\|X\|_\psi = \inf\left\{K > 0 : \mathbb{E}[\psi(|X|/K)] \leq 1\right\}.$$

Consider any $\alpha \geq 1$, and $\psi_\alpha(x) = e^{x^\alpha} - 1$. Show that $\|X\|_{\psi_\alpha} < \infty$ if and only if there exists $C, c > 0$ such that $\mathbb{P}[|X| \geq t] \leq Ce^{-ct^\alpha}$.

4. (BLM 2.27) In the setting of Problem 3, suppose that $X_1, \ldots, X_n$ are i.i.d. with $\mathbb{E}X_i = 0$ and $\|X_i\|_{\psi_\alpha} \leq K < \infty$, where $\alpha \in (1, 2)$. Prove that there exist constants $C, c > 0$ depending only on K, α such that

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^n X_i \geq t\right] \leq C\exp\left(-cn\min\left(t^2, t^\alpha\right)\right).$$

[Hint: Show $\log \mathbb{E}[e^{\lambda X_i}] \leq C\lambda^2$ for $\lambda \in [0, 1]$, and $\log \mathbb{E}[e^{\lambda X_i}] \leq C\lambda^{\alpha/(\alpha-1)}$ for $\lambda > 1$.]