S&DS 602: Homework 5

Due Wednesday, October 2 at 2PM, via Gradescope

1. Let $X \in \mathbb{R}^{n\times n}$ be a matrix with i.i.d. entries $X_{ij} \sim \mathcal{N}(0, \frac{1}{n})$, let $f : \{+1, -1\}^n \to \mathbb{R}$ be arbitrary, and consider

$$Z = \frac{1}{n} \log \sum_{\sigma \in \{+1,-1\}^n} \exp\Big(\sigma^\top X \sigma + f(\sigma)\Big).$$

Show that for any $t \geq 0$,

$$\mathbb{P}[|Z - \mathbb{E}Z| \geq t] \leq 2e^{-\frac{nt^2}{2}}.$$

2. (Lipschitz extension) (a) Let $S \subseteq \mathbb{R}^n$, $L > 0$, and let $f : \mathbb{R}^n \to \mathbb{R}$ satisfy

$$\|f(x) - f(y)\|_2 \leq L\|x - y\|_2 \text{ for all } x, y \in S.$$

Consider the function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ given by

$$\bar{f}(x) = \inf_{y \in S} f(y) + L\|x - y\|_2.$$

Show that $\bar{f}(x) = f(x)$ for all $x \in S$, and $\bar{f}$ is L-Lipschitz on all of $\mathbb{R}^n$.

(b) Let $S \subseteq \mathbb{R}^n$, $L > 0$, and $f : \mathbb{R}^n \to \mathbb{R}$ be as above. Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(0, 1)$, and suppose in addition

$$\mathbb{E}f(X_1, \ldots, X_n)^2 \leq M, \quad \mathbb{E}\bar{f}(X_1, \ldots, X_n)^2 \leq M, \quad \mathbb{P}[(X_1, \ldots, X_n) \notin S] \leq \delta.$$

Show that for any $t \geq 0$,

$$\mathbb{P}\Big[|f(X_1, \ldots, X_n) - \mathbb{E}f(X_1, \ldots, X_n)| \geq t + 2\sqrt{M\delta}\Big] \leq 2e^{-\frac{t^2}{2L^2}} + \delta.$$

3. (Holley-Stroock perturbation) (a) Let P be a probability distribution on $\mathbb{R}$ satisfying the log-Sobolev inequality, for all continuously-differentiable $f : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{Ent}_{X\sim P}[f(X)^2] \leq C_{\mathrm{LSI}} \cdot \mathbb{E}_{X\sim P}[f'(X)^2].$$

Let $Q \ll P$ be such that $\epsilon \leq \frac{dQ}{dP}(x) \leq \omega$ for all $x \in \mathbb{R}$ and some $\epsilon, \omega > 0$. Show that Q satisfies the log-Sobolev inequality, for all continuously-differentiable $f : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{Ent}_{X\sim Q}[f(X)^2] \leq \frac{\omega C_{\mathrm{LSI}}}{\epsilon} \cdot \mathbb{E}_{X\sim Q}[f'(X)^2].$$

[Hint: Use the identity $\operatorname{Ent} Z = \inf_{t \geq 0} \mathbb{E}[Z \log Z - Z \log t - Z + t]$.]

(b) Suppose Q has density $q(x) = e^{-V(x)} / \int e^{-V(t)} dt$ where $\frac{x^2}{2} + a \leq V(x) \leq \frac{x^2}{2} + b$ for all $x \in \mathbb{R}$. Show that for all continuously-differentiable $f : \mathbb{R} \to \mathbb{R}$,

$$\operatorname{Ent}_{X \sim Q}[f(X)^2] \leq 2e^{2(b-a)}\, \mathbb{E}_{X \sim Q}[f'(X)^2].$$

4. Suppose Z is a random variable such that, for some $\sigma^2, b > 0$,

$$\operatorname{Ent} e^{\lambda Z} \leq \lambda^2 \Big[b\, \mathbb{E} Z e^{\lambda Z} + (\sigma^2 - b\, \mathbb{E} Z) \mathbb{E} e^{\lambda Z} \Big] \text{ for all } \lambda \in [0, \tfrac{1}{b}).$$

Show that

$$\log \mathbb{E}[e^{\lambda (Z - \mathbb{E} Z)}] \leq \frac{\lambda^2 \sigma^2}{1 - \lambda b} \text{ for all } \lambda \in [0, \tfrac{1}{b})$$

and hence $Z - \mathbb{E} Z$ is $(4\sigma^2, 2b)$-subexponential.

[Hint: Consider first $\mathbb{E} Z = 0$, and use that the given inequality is equivalent to $\frac{d}{d\lambda}[\frac{1}{\lambda} \log F(\lambda)] \leq \sigma^2 + b \cdot \frac{d}{d\lambda}[\log F(\lambda)]$ where $F(\lambda) = \mathbb{E} e^{\lambda Z}$.]