S&DS 602: Homework 7

Due Wednesday, October 23 at 2PM, via Gradescope

1. (Vershynin 4.2.9) Let T be a subset of a metric space (S, d). Define the exterior covering number
$$N^{\text{ext}}(T, d, \varepsilon) = \inf\Big\{|\mathcal{N}| : \mathcal{N} \subseteq S \text{ is an } \varepsilon\text{-net of } T\Big\},$$
where the points of $\mathcal{N}$ may be chosen in S rather than in T. Show that
$$N^{\text{ext}}(T, d, \varepsilon) \leq N(T, d, \varepsilon) \leq N^{\text{ext}}(T, d, \varepsilon/2).$$

2. (Vershynin 4.2.16) Let $T = \{0, 1\}^n$, equipped with the Hamming metric $d(s, t) = \sum_{i=1}^n \mathbf{1}\{s_i \neq t_i\}$. Let $N(T, d, m)$ and $D(T, d, m)$ be the covering and packing numbers of T. Show that for every integer $m \in \{1, \ldots, n\}$,
$$\frac{2^n}{\sum_{k=0}^m \binom{n}{k}} \leq N(T, d, m) \leq D(T, d, m) \leq \frac{2^n}{\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}$$
[Hint: Mimic the volume argument from Lecture 7 for the covering number of the Euclidean ball B^n.]

3. (van Handel 5.8) (a) Let $\mathcal{F}$ be the class of functions $f : [0, 1]^d \to [0, 1]$ satisfying the Lipschitz condition $|f(x) - f(y)| \leq \|x - y\|_2$. Show that for some $C > 0$ depending only on d, and for any $\varepsilon \in (0, 1)$,
$$N(\mathcal{F}, \|\cdot\|_\infty, \varepsilon) \leq C^{1/\varepsilon^d}$$
(b) Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $[0, 1]^d$, and consider
$$W = \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n f(X_i) - \mathbb{E}f(X_i).$$
Show that for a constant $C > 0$ depending only on d,
$$\mathbb{E}W \leq Cn^{-\frac{1}{d+2}}.$$

4. Let $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ be a function whose gradient $\nabla f(X) = (\partial_{x_{ij}} f(X))_{1\leq i\leq n,\, 1\leq j\leq m} \in \mathbb{R}^{n\times m}$ satisfies, for a constant $L > 0$ and all $X \in \mathbb{R}^{n\times m}$,

$$\|\nabla f(X)\|_F \leq L\|X\|_{\mathrm{op}}.$$

If $X \in \mathbb{R}^{n\times m}$ has i.i.d. $\mathcal{N}(0,1)$ entries, show that for some universal constants $C, c > 0$ and all $t \geq 0$,

$$\mathbb{P}[|f(X) - \mathbb{E}f(X)| \geq t] \leq C(e^{-\frac{ct^2}{L^2(n+m)}} + e^{-(n+m)}).$$

[Hint: Apply Problem 2 of Homework 5 with an appropriate choice of the set S.]