S&DS 602: Homework 8

Due Wednesday, October 30 at 2PM, via Gradescope

1. (van Handel 5.10) Let $\psi : [0, \infty) \to [0, \infty)$ be convex with $\psi(0) = \psi'(0) = 0$, and set $\psi^*(x) = \sup_{\lambda \geq 0} \lambda x - \psi(\lambda)$. Suppose $\{X_t\}_{t \in T}$ is a separable, mean-zero process with

$$\log \mathbb{E}[e^{\lambda(X_t - X_s)/d(t,s)}] \leq \psi(\lambda) \text{ for all } t, s \in T \text{ and } \lambda \geq 0.$$

Apply the chaining argument to show, for a universal constant $C > 0$,

$$\mathbb{E} \sup_{t \in T} X_t \leq C \int_0^\infty \psi^{*-1}(2 \log N(T, d, \varepsilon)) d\varepsilon.$$

2. (van Handel 5.11) Recall the setting of Homework 7, Problem 3: Let $\mathcal{F}$ be the class of functions $f : [0, 1]^d \to [0, 1]$ satisfying the Lipschitz condition $|f(x) - f(y)| \leq \|x - y\|_2$. Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $[0, 1]^d$, and consider

$$W = \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n f(X_i) - \mathbb{E} f(X_i).$$

(a) Show that for any $d \geq 2$ and a constant $C > 0$ depending only on d,

$$\mathbb{E} W \leq C \cdot \begin{cases} n^{-1/2} \log n & \text{if } d = 2 \\ n^{-1/d} & \text{if } d \geq 3. \end{cases}$$

You may use without proof the following extension of the chaining result from lecture: If $\{X_t\}_{t \in T}$ is a separable mean-zero sub-gaussian process with respect to $d(t, s)$, then there is a universal constant $C > 0$ such that for any $\delta > 0$,

$$\mathbb{E} \sup_{t \in T} X_t \leq C \int_\delta^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon + 2\mathbb{E} \sup_{t, s \in T : \, d(t,s) \leq \delta} |X_t - X_s|.$$

(b) Suppose that the law of X_i has bounded density on $[0, 1]^d$. Show that for some $c > 0$ and all $x \in [0, 1]^d$,

$$\mathbb{E} \min_{i=1}^n \|x - X_i\|_2 \geq c n^{-1/d},$$

and hence $\mathbb{E} W \geq c n^{-1/d}$.

3. (a) Let $\{X_n\}_{n\geq 1}$ be $\mathcal{N}(0,1)$ random variables (not necessarily independent), and $\{a_n\}_{n\geq 1}$ a sequence of positive numbers. Show that if $a_n > c\sqrt{\log n}$ for some $c > 0$ and all $n \geq 1$, then

$$\mathbb{E} \sup_{n\geq 1} \frac{X_n}{a_n} < \infty.$$

(b) Suppose, in addition, $\{X_n\}_{n\geq 1}$ are independent. Show that conversely, if $a_n/\sqrt{\log n} \to 0$ as $n \to \infty$, then $\sup_{n\geq 1} X_n/a_n = \infty$ with probability 1.

4. (Vershynin 8.1.12) Let $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$, and consider the set

$$T = \left\{ \frac{e_k}{\sqrt{1+\log k}} : k = 1, \ldots, n \right\}.$$

Let $g \sim \mathcal{N}(0, I)$ be a standard Gaussian vector in $\mathbb{R}^n$. Show that for a universal constant $C > 0$,

$$\mathbb{E} \sup_{t \in T} g^\top t \leq C.$$

However, as $n \to \infty$,

$$\int_0^\infty \sqrt{\log N(T, \|\cdot\|_2, \varepsilon)}\, d\varepsilon \to \infty.$$

(Thus a direct application of Dudley's integral upper bound is not tight for large n.)