S&DS 602: Homework 10

Due Wednesday, November 13 at 2PM, via Gradescope

1. Let $X \in \mathbb{R}^{n \times n}$ be a symmetric random matrix whose entries above the diagonal are i.i.d. $\mathcal{N}(0,1)$, and whose entries on the diagonal are i.i.d. $\mathcal{N}(0,2)$. Let $\lambda_{\max}(X)$ be its largest eigenvalue. Apply the Sudakov-Fernique inequality to show that

$$\mathbb{E}[\lambda_{\max}(X)] \le 2\sqrt{n}.$$

Show that this implies, for a universal constant $c > 0$ and any $t \ge 0$,

$$\mathbb{P}[\lambda_{\max}(X) \ge 2\sqrt{n} + t] \le e^{-ct^2}.$$

[Hint: $\lambda_{\max}(X) = \sup_{u \in S^{n-1}} u^\top X u$. Compare with $\sup_{u \in S^{n-1}} 2u^\top g$ where $g \sim \mathcal{N}(0, I)$.]

The remaining problems relate to the following extension of the Slepian-Sudakov-Fernique inequalities due to Gordon:

Theorem. (Gordon's inequalities) Let $\{X_{ut}\}_{(u,t) \in U \times T}$ and $\{Y_{ut}\}_{(u,t) \in U \times T}$ be two separable mean-zero Gaussian processes.

(a) Suppose, for all $u \neq v \in U$ and all $t, s \in T$, that

$$\mathbb{E}(X_{ut} - X_{us})^2 \le \mathbb{E}(Y_{ut} - Y_{us})^2, \qquad \mathbb{E}(X_{ut} - X_{vs})^2 \ge \mathbb{E}(Y_{ut} - Y_{vs})^2.$$

Then

$$\mathbb{E}\left[\inf_{u \in U} \sup_{t \in T} X_{ut}\right] \le \mathbb{E}\left[\inf_{u \in U} \sup_{t \in T} Y_{ut}\right].$$

(b) If, in addition, $\mathbb{E}X_{ut}^2 = \mathbb{E}Y_{ut}^2$ for all $u \in U$ and $t \in T$, then for any $\tau \in \mathbb{R}$,

$$\mathbb{P}\left[\inf_{u \in U} \sup_{t \in T} X_{ut} \ge \tau\right] \le \mathbb{P}\left[\inf_{u \in U} \sup_{t \in T} Y_{ut} \ge \tau\right].$$

Problem 2 will ask you to show part (b). You may use both parts (a) and (b) in the remaining Problems 3 and 4.

2. (Vershynin 7.2.14) Prove part (b) of the above theorem.

   [Hint: The proof is similar to that of Slepian's inequality. Consider first the case where U and T are finite sets, and apply Gaussian interpolation to the function

$$\prod_{u \in U} \left(1 - \prod_{t \in T} \varphi(X_{ut})\right)$$

   where $\varphi(x)$ is a smooth approximation to $\mathbf{1}\{x < \tau\}$.]

3. (Vershynin 7.3.4) Suppose $X \in \mathbb{R}^{n \times m}$ has i.i.d. $\mathcal{N}(0,1)$ entries, and $n \geq m$. Let $s_{\min}(X)$ be its smallest singular value. Apply Gordon's inequality to show

$$\mathbb{E}s_{\min}(X) \geq \sqrt{n} - \sqrt{m}.$$

   Show that this implies, for a universal constant $c > 0$ and any $t \geq 0$,

$$\mathbb{P}[s_{\min}(X) \leq \sqrt{n} - \sqrt{m} - t] \leq e^{-ct^2}.$$

   [Hint: $s_{\min}(X) = \inf_{u \in S^{m-1}} \|Xu\|_2 = \inf_{u \in S^{m-1}} \sup_{t \in S^{n-1}} t^\top X u$.]

4. (Convex Gaussian min-max theorem) (a) Let $X \in \mathbb{R}^{n \times m}$, $g \in \mathbb{R}^n$, and $h \in \mathbb{R}^m$ be independent with i.i.d. $\mathcal{N}(0,1)$ entries. Let $U \subset \mathbb{R}^n$, $T \subset \mathbb{R}^m$, and define

$$\Phi(X) = \inf_{u \in U} \sup_{t \in T} u^\top X t$$
$$\phi(g,h) = \inf_{u \in U} \sup_{t \in T} \|u\|_2 g^\top t + \|t\|_2 h^\top u.$$

   Show that for any $\tau \in \mathbb{R}$,

$$\mathbb{P}[\Phi(X) < \tau] \leq 2\,\mathbb{P}[\phi(g,h) < \tau].$$

   [Hint: Let $z \sim \mathcal{N}(0,1)$ be independent of X, g, h, and consider

$$\tilde{\Phi}(X,z) = \inf_{u \in U} \sup_{t \in T} u^\top X t + z\|u\|_2\|t\|_2.$$

   Show first that $\mathbb{P}[\Phi(X) < \tau] \leq \mathbb{P}[\tilde{\Phi}(X,z) < \tau \mid z \leq 0] \leq 2\,\mathbb{P}[\tilde{\Phi}(X,z) < \tau]$.]

   (b) Suppose, in addition, that U and T are compact and convex sets. Show that also for any $\tau \in \mathbb{R}$,

$$\mathbb{P}[\Phi(X) > \tau] \leq 2\,\mathbb{P}[\phi(g,h) > \tau],$$

   and hence for any $c \in \mathbb{R}$ and $\tau > 0$,

$$\mathbb{P}[|\Phi(X) - c| > \tau] \leq 2\,\mathbb{P}[|\phi(g,h) - c| > \tau].$$

   [Hint: For any continuous $f : U \times T \to \mathbb{R}$ that is convex in u and concave in t,

$$\sup_{t \in T} \inf_{u \in U} f(u,t) = \inf_{u \in U} \sup_{t \in T} f(u,t).$$

   (You may use this without proof.) Apply this and part (a) with $-\Phi(X)$.]