S&DS 602: Homework 11

Due Wednesday, November 20 at 2PM, via Gradescope

1. Let T be a polytope in $\mathbb{R}^n$ with N vertices (i.e. the convex hull of N points in $\mathbb{R}^n$) which is contained in the unit ball B^n. Let d be the Euclidean metric on $\mathbb{R}^n$.

   (a) Show the covering number bound, for a universal constant $C > 0$,

   $$N(T, d, \varepsilon) \leq N^{C/\varepsilon^2}.$$

   [Hint: Apply the Sudakov lower bound $\mathbb{E}\sup_{t \in T} g^\top t \geq c\varepsilon\sqrt{\log N(T, d, \varepsilon)}$.]

   (b) Using part (a), show that the volume of T satisfies, for a universal constant $C > 0$,

   $$\text{Vol}(T) \leq \text{Vol}(B^n)\Big(\frac{C \log N}{n}\Big)^{Cn}.$$

2. (van Handel 6.5) Let $a_1, \ldots, a_n > 0$ be fixed positive numbers, $g_1, \ldots, g_n \overset{iid}{\sim} \mathcal{N}(0, 1)$, and $T = \{+1, -1\}^n$. Consider the Gaussian process $\{X_t\}_{t \in T}$ given by

   $$X_t = \sum_{k=1}^{n} g_k t_k a_k.$$

   (a) Show that

   $$\mathbb{E}\sup_{t \in T} X_t = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{n} a_k$$

   (b) Explain why $\{X_t\}_{t \in T}$ is a stationary process with respect to the canonical metric $d(s, t) = \sqrt{\mathbb{E}(X_t - X_s)^2}$, and hence for some universal constants $C, c > 0$,

   $$c \int_0^\infty \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon \leq \mathbb{E}\sup_{t \in T} X_t \leq C \int_0^\infty \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon$$

   (c) Consider $a_k = 1/k$ for each $k = 1, \ldots, n$. Show that for universal constants $c, c' > 0$ and any $n \geq 1$,

   $$\sup_{\varepsilon > 0} \varepsilon\sqrt{\log N(T, d, \varepsilon)} \leq c$$

   but $\sum_{k=1}^{n} a_k \geq c' \log n$. [Thus the Dudley integral bound is tight in this example, but Sudakov's lower bound is not.]

3. (van Handel 6.8) Let (T, d) be any metric space. Recall the generic chaining functional, for a universal constant $\alpha \in (0, 1)$,

$$\gamma(T) = \inf_{(\mathcal{A},\ell)} \sup_{t \in T} \sum_{k \in \mathbb{Z}} \alpha^k \sqrt{\log \ell(A_k(t))}$$

where the infimum is over all labeled nets $(\mathcal{A}, \ell)$ of (T, d). Construct a suitable labeled net $(\mathcal{A}, \ell)$ to show directly that for a universal constant $C > 0$,

$$\gamma(T) \leq C \int_0^\infty \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon.$$

4. (van Handel 6.9) Recall the example from Homework 8, Problem 4: Let $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$, and consider

$$T = \left\{ \frac{e_k}{\sqrt{1 + \log k}} : k = 1, \ldots, n \right\}$$

with the canonical metric $d(t, s) = \|t - s\|_2$. Let

$$\gamma(T) = \inf_{(\mathcal{A},\ell)} \sup_{t \in T} \sum_{k \in \mathbb{Z}} \alpha^k \sqrt{\log \ell(A_k(t))}$$

be the generic chaining functional. Construct a suitable labeled net $(\mathcal{A}, \ell)$ to show that there exists a universal constant $C > 0$ for which $\gamma(T) \leq C$.