S&DS 602: Homework 12

Due Wednesday, December 4 at 2PM, via Gradescope

1. (Vershynin 7.6.9) Let $g \sim \mathcal{N}(0, I)$ be a standard Gaussian vector in $\mathbb{R}^n$, and $T \subset \mathbb{R}^n$ a bounded set. Define

$$w(T) = \mathbb{E} \sup_{t \in T} g^\top t, \qquad \tilde{w}(T) = \mathbb{E} \sup_{t \in T} |g^\top t|.$$

   Show that for universal constants $C, c > 0$ and any point in $t_0 \in T$,

$$c[w(T) + \|t_0\|_2] \le \tilde{w}(T) \le C[w(T) + \|t_0\|_2].$$

   In particular, if T contains 0, then $w(T)$ and $\tilde{w}(T)$ are equivalent up to constant factors. [Hint: Recall the relations from lecture $\sup_{s,t \in T} |g^\top (t-s)| = \sup_{s,t \in T} g^\top (t-s) = 2w(T)$.]

2. (Vershynin 8.7.2) Let $A \in \mathbb{R}^{n \times m}$ be a random matrix with $\mathcal{N}(0, 1)$ entries. Let $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ be bounded sets. Show that the reverse of Chevet's inequality holds: For a universal constant $c > 0$,

$$\mathbb{E} \sup_{u \in S,\, v \in T} u^\top A v \ge c\Big[w(T)\,\mathrm{rad}(S) + w(S)\,\mathrm{rad}(T)\Big].$$

3. (Vershynin 11.3.2) Let $V \subset \mathbb{R}^n$ be a closed bounded set, $\mathrm{conv}(V)$ its convex hull, and B^n and S^{n-1} the Euclidean unit ball and unit sphere in $\mathbb{R}^n$, respectively.

   (a) Show that $\mathrm{conv}(V) = B^n$ if and only if

$$\sup_{x \in V} x^\top y = \|y\|_2 \text{ for all } y \in S^{n-1}.$$

   (b) Let $r_-, r_+ \ge 0$. Show that $r_- B^n \subseteq \mathrm{conv}(V) \subseteq r_+ B^n$ if and only if

$$r_- \|y\|_2 \le \sup_{x \in V} x^\top y \le r_+ \|y\|_2 \text{ for all } y \in S^{n-1}.$$

4. (Vershynin 11.3.7) Let $g_1, \ldots, g_m \sim \mathcal{N}(0, I)$ be i.i.d. random Gaussian vectors in $\mathbb{R}^n$. Applying the Dvoretzky-Milman Theorem, show that if $m \ge \exp(Cn)$ for a large enough absolute constant $C > 0$, then the convex hull of $\{g_1, \ldots, g_m\}$ is approximately a Euclidean ball of radius $\sim \sqrt{2 \log m}$.