Gaussian integration by parts method

Let $X \in \mathbb{R}^{n\times n}$ be a Wigner matrix. Let $G_n(z) = (X - zI)^{-1}$ be its resolvent.

Recall the Stieltjes transform of the ESD of X,

$$m_n(z) = \frac{1}{n} \operatorname{Tr} G_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i(X) - z}.$$

For any $z \in \mathbb{C}^+$:

$$I = (X - zI)\cdot(X - zI)^{-1} = (X - zI) G_n(z).$$

Then $G_n(z) = \frac{1}{z}(X G_n(z) - I)$

$$m_n(z) = \frac{1}{nz} \operatorname{Tr}(X G_n(z) - I)$$

$$= \frac{1}{nz} \sum_{i,j=1}^{n} X_{ij} G_n(z)_{ji} - \frac{1}{z}.$$

Here, $G_n(z)_{ji}$ is a function of X_{ij} and other independent variables.

If X_{ij} were Gaussian, integration-by-parts yields a formula for $\mathbb{E}[X_{ij} G_n(z)_{ji}]$, which may be used to derive a recursion for $\mathbb{E}[m_n(z)]$.

Lemma (integration by parts): Let $\xi \sim N(0,1)$. If $f : \mathbb{R} \to \mathbb{C}$ is differentiable, with $|f(x) \cdot e^{-\frac{x^2}{2}}| \to 0$ as $x \to \pm\infty$, then

$$\mathbb{E}[\xi \cdot f(\xi)] = \mathbb{E}[f'(\xi)].$$

Proof: $\mathbb{E}[\xi \cdot f(\xi)] = \int_{-\infty}^{\infty} \underbrace{f(x)}_{u} \cdot \underbrace{x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{dv}$

$$= \left[-f(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f'(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \mathbb{E}[f'(\xi)].$$

Cor: Let $\xi \sim N(0, \sigma^2)$. Under the preceding conditions,

$$\mathbb{E}[\xi f(\xi)] = \sigma^2 \cdot \mathbb{E}[f'(\xi)].$$

Proof: Write $\xi = \sigma \cdot \zeta$. Apply the preceding to $\zeta \mapsto f(\sigma\zeta)$. Then

$$\mathbb{E}[\xi f(\xi)] = \sigma \mathbb{E}[\zeta \cdot f(\sigma\zeta)] = \sigma^2 \mathbb{E}[f'(\sigma\zeta)] = \sigma^2 \mathbb{E}[f'(\xi)].$$

Lemma: Let $X \sim GOE(n)$, i.e. $X_{ij} = \frac{1}{\sqrt{n}} Z_{ij}$ where

$$Z_{ij} \sim \begin{cases} N(0,1) & i \neq j \\ N(0,2) & i = j. \end{cases}$$

Let $G_n(z) = (X - zI)^{-1}$, $m_n(z) = \frac{1}{n} \operatorname{Tr} G_n(z)$. Then for all $z \in \mathbb{C}^+$,

$$\mathbb{E}[m_n(z)] = -\frac{1}{z} - \frac{1}{z}\mathbb{E}[m_n(z)^2] - \frac{1}{n^2 z}\mathbb{E}[\operatorname{Tr} G_n(z)^2].$$

Proof: Write $X = \frac{1}{\sqrt{2n}}(W + W^T)$, where $W \in \mathbb{R}^{n\times n}$ is asymmetric with $N(0,1)$ entries. Note that

- $X - zI = \frac{1}{\sqrt{2n}}\left(\sum_{i,j=1}^{n} W_{ij}\Delta_{ij} + W_{ij}\Delta_{ji}\right) - zI$

  where $\Delta_{ij} \in \mathbb{R}^{n\times n}$ has (i,j) entry 1 and remaining entries 0.

  Then $\frac{\partial}{\partial W_{ij}}(X - zI) = \frac{1}{\sqrt{2n}}(\Delta_{ij} + \Delta_{ji}) \in \mathbb{R}^{n\times n}$.

- If $A(t) \in \mathbb{C}^{n\times n}$ has entries differentiable in the parameter $t \in \mathbb{R}$, and is invertible at t, then

$$\Rightarrow \quad I = A(t)\cdot A(t)^{-1}$$

$$\Rightarrow \quad 0 = \frac{d}{dt}\left(A(t)\cdot A(t)^{-1}\right)$$

$$= A'(t)\cdot A(t)^{-1} + A(t)\cdot \frac{d}{dt}\left(A(t)^{-1}\right).$$

So $\frac{d}{dt}\left(A(t)^{-1}\right) = -A(t)^{-1}\cdot A'(t)\cdot A(t)^{-1}.$

Then, applying to $A(W_{ij}) = X - zI$,

$$\frac{\partial}{\partial W_{ij}} (X-zI)^{-1} = -(X-zI)^{-1}\left(\frac{1}{\sqrt{2n}}(\Delta_{ij}+\Delta_{ji})\right)(X-zI)^{-1}.$$

So $$\frac{\partial}{\partial W_{ij}} G_n(z) = -\frac{1}{\sqrt{2n}} G_n(z)(\Delta_{ij}+\Delta_{ji})G_n(z).$$

$$\frac{\partial}{\partial W_{ij}} G_n(z)_{ij} = -\frac{1}{\sqrt{2n}}\left(G_n(z)_{ii}G_n(z)_{jj} + G_n(z)_{ij}G_n(z)_{ij}\right).$$

$$\frac{\partial}{\partial W_{ij}} G_n(z)_{ji} = \frac{\partial}{\partial W_{ij}} G_n(z)_{ij}$$ because G_n is symmetric.

Applying integration by parts,

$$\mathbb{E}[X_{ij}G_n(z)_{ji}] = \frac{1}{\sqrt{2n}}\mathbb{E}\left[(W_{ij}+W_{ji})G_n(z)_{ji}\right]$$

$$= \frac{1}{\sqrt{2n}}\mathbb{E}\left[\frac{\partial}{\partial W_{ij}}G_n(z)_{ji} + \frac{\partial}{\partial W_{ji}}G_n(z)_{ji}\right]$$

$$= -\frac{1}{n}\left(\mathbb{E}[G_n(z)_{ii}G_n(z)_{jj}] + \mathbb{E}[G_n(z)_{ji}^2]\right).$$

Then $$\mathbb{E}[m_n(z)] = -\frac{1}{z} + \frac{1}{nz}\sum_{i,j=1}^n \mathbb{E}[X_{ij}G_n(z)_{ji}]$$

$$= -\frac{1}{z} - \frac{1}{n^2 z}\mathbb{E}\left[\left(\sum_{i=1}^n G_n(z)_{ii}\right)^2\right] - \frac{1}{n^2 z}\mathbb{E}\left[\mathrm{Tr}\, G_n(z)^T G_n(z)\right].$$

$$= -\frac{1}{z} - \frac{1}{z}\mathbb{E}[m_n(z)^2] - \frac{1}{n^2 z}\mathbb{E}[\mathrm{Tr}\, G_n(z)^2].$$

Prop: $\left|\frac{1}{n}\mathrm{Tr}\, G_n(z)^2\right| \le \frac{1}{(\mathrm{Im}\, z)^2}$, for $z \in \mathbb{C}^+$.

Proof: Let $X = \sum_{i=1}^n \lambda_i v_i v_i^T$ be the spectral decomposition of X.

Then $(X-zI)^{-1} = \sum_{i=1}^n \frac{1}{\lambda_i - z} v_i v_i^T$ is the spectral decomposition of $G_n(z)$,

$(X-zI)^{-2} = \sum_{i=1}^n \frac{1}{(\lambda_i - z)^2} v_i v_i^T$ that of $G_n(z)^2$.

So $$\left|\frac{1}{n}\mathrm{Tr}\, G_n(z)^2\right| = \left|\frac{1}{n}\sum_{i=1}^n \frac{1}{(\lambda_i - z)^2}\right| \le \frac{1}{n}\sum_{i=1}^n \frac{1}{|\lambda_i - z|^2} \le \frac{1}{(\mathrm{Im}\, z)^2}.$$

Prop: For constants $C, c > 0$, and any $z \in \mathbb{C}^+$,

$$\mathbb{P}\left[|m_n(z) - \mathbb{E}m_n(z)| \ge t\right] \le 4e^{-ct^2(\mathrm{Im}\, z)^4 n^2}$$

and $$\left|\mathbb{E}[m_n(z)^2] - \mathbb{E}[m_n(z)]^2\right| \le \frac{C}{n(\mathrm{Im}\, z)^3}.$$

Proof: Recall for $x, x' \in \mathbb{R}$, $\left|\frac{1}{x-z} - \frac{1}{x'-z}\right| \le \frac{|x-x'|}{(\mathrm{Im}\, z)^2}$. So

$x \mapsto \mathrm{Re}\,\frac{1}{x-z}$ and $x \mapsto \mathrm{Im}\,\frac{1}{x-z}$ are both $\frac{1}{(\mathrm{Im}\, z)^2}$-Lipschitz.

From concentration of spectral measure,

$$\mathbb{P}\left[\left|\int \mathrm{Re}\,\frac{1}{x-z}\,d\mu_n(x) - \mathbb{E}\int \mathrm{Re}\,\frac{1}{x-z}\,d\mu_n(x)\right| \ge \frac{t}{2}\right] \le 2e^{-ct^2(\mathrm{Im}\, z)^4 n^2}.$$

Similarly for $\mathrm{Im}\,\frac{1}{x-z}$, and combining yields the bound for $|m_n - \mathbb{E}m_n|$.

For the second statement, let $\bar{m}_n(z) = \mathbb{E}m_n(z)$.

$$\left|\mathbb{E}[m_n(z)^2] - \bar{m}_n(z)^2\right| = \left|\mathbb{E}\left[(m_n(z)-\bar{m}_n(z))(m_n(z)+\bar{m}_n(z))\right]\right|$$

$$\le \mathbb{E}\left[|m_n(z)-\bar{m}_n(z)|\cdot|m_n(z)+\bar{m}_n(z)|\right]$$

$$\le \frac{2}{\mathrm{Im}\, z}\mathbb{E}\left[|m_n(z)-\bar{m}_n(z)|\right].$$

(We've used $|m_n(z)| \le \frac{1}{\mathrm{Im}\, z}$.)

For any non-negative random variable Z,

$$\mathbb{E}[Z] = \mathbb{E}\left[\int_0^Z dt\right] = \mathbb{E}\left[\int_0^\infty \mathbb{1}\{Z \ge t\}\,dt\right] = \int_0^\infty \mathbb{P}[Z \ge t]\,dt.$$

So $$\mathbb{E}\left[|m_n(z)-\bar{m}_n(z)|\right] = \int_0^\infty \mathbb{P}\left[|m_n(z)-\bar{m}_n(z)| \ge t\right]dt$$

$$\le 4\int_0^\infty e^{-ct^2(\mathrm{Im}\, z)^4 n^2}\,dt$$

$$= 2\int_{-\infty}^\infty e^{-ct^2(\mathrm{Im}\, z)^4 n^2}\,dt \le \frac{C}{n(\mathrm{Im}\, z)^2}.$$

Proof #3 of semicircle law, GOE case:

By the preceding results,

$$\mathbb{E}[m_n(z)] = -\frac{1}{z} - \frac{1}{z}\mathbb{E}[m_n(z)]^2 + r_n$$

where $|r_n| \le \frac{C}{n}\left(\frac{1}{(\operatorname{Im} z)^3} + \frac{1}{(\operatorname{Im} z)^4}\right)$. Let $m(z)$ be the root of

$$m(z) = -\frac{1}{z} - \frac{1}{z}m(z)^2 \iff m(z)^2 + zm(z) + 1 = 0$$

with positive imaginary part. Since $\mathbb{E}[m_n(z)]$ has positive imaginary part, this implies $|\mathbb{E}m_n(z) - m(z)| \to 0$.

From the preceding proposition, $|m_n(z) - \mathbb{E}m_n(z)| \to 0$ a.s. by Borel-Cantelli.

So $m_n(z) \to m(z)$ a.s.

This may be extended to other models. For example:

Let $X \sim GOE(n)$. Let $A \in \mathbb{R}^{n\times n}$ be deterministic, symmetric, with the ESD of A converging to μ_A weakly. What is the limit ESD of $X+A$?

Denote $G_n(z) = (X + A - zI)^{-1}$, $m_n(z) = \frac{1}{n}\operatorname{Tr} G_n(z)$.

$$I = (X + A - zI)\cdot G_n(z)$$

$$\Rightarrow (zI - A)\cdot G_n(z) = XG_n(z) - I.$$

We apply integration by parts without taking the trace:

$$\mathbb{E}[(XG_n(z))_{ij}] = \sum_{k=1}^n \mathbb{E}[X_{ik}G_n(z)_{kj}]$$

$$= -\frac{1}{n}\sum_{k=1}^n \mathbb{E}[(G_n(z)(\Delta_{ik} + \Delta_{ki})G_n(z))_{kj}]$$

$$= -\frac{1}{n}\sum_{k=1}^n \mathbb{E}[G_n(z)_{ki}G_n(z)_{kj} + G_n(z)_{kk}G_n(z)_{ij}]$$

$$= -\frac{1}{n}\mathbb{E}[(G_n(z)^2)_{ij}] - \mathbb{E}[m_n(z)\cdot G_n(z)_{ij}].$$

So $\mathbb{E}[XG_n(z)] = -\frac{1}{n}\mathbb{E}[G_n(z)^2] - \mathbb{E}[m_n(z)G_n(z)]$.

Writing $G_n(z) = \sum_{i=1}^n \frac{1}{\lambda_i - z}v_iv_i^T$, $\|G_n(z)\| = \sup_i \frac{1}{|\lambda_i - z|} \le \frac{1}{\operatorname{Im} z}$.

Then $\|\frac{1}{n}\mathbb{E}[G_n(z)^2]\| \le \frac{1}{n(\operatorname{Im} z)^2}$. Same argument shows concentration of $m_n(z)$.

Then $\|\mathbb{E}[(m_n(z) - \bar{m}_n(z))G_n(z)]\|$

$$\le \mathbb{E}[|m_n(z) - \bar{m}_n(z)|]\cdot\frac{1}{\operatorname{Im} z} \le \frac{C}{n(\operatorname{Im} z)^3}.$$

Thus $\mathbb{E}[XG_n(z)] = -\bar{m}_n(z)\cdot\mathbb{E}[G_n(z)] + R_n$,

where $\|R_n\| \le \frac{1}{n}\cdot\operatorname{poly}\left(\frac{1}{\operatorname{Im} z}\right)$.

We get $(zI - A)\cdot\mathbb{E}[G_n(z)] = -\bar{m}_n(z)\cdot\mathbb{E}[G_n(z)] - I + R_n$

$$\Rightarrow \mathbb{E}[G_n(z)] = (zI + \bar{m}_n(z)I - A)^{-1}(-I + R_n)$$

$$= (A - (z + \bar{m}_n(z))I)^{-1} + \tilde{R}_n$$

where $\|\tilde{R}_n\| \le \frac{1}{n}\operatorname{poly}\left(\frac{1}{\operatorname{Im} z}\right)$ also. Now taking $\frac{1}{n}\operatorname{Tr}$,

$$\bar{m}_n(z) = \frac{1}{n}\operatorname{Tr}(A - (z + \bar{m}_n(z))I)^{-1} + O\left(\frac{1}{n}\right)$$

$$\to m_A(z + \bar{m}_n(z)),$$

where m_A is the Stieltjes transform of μ_A (the limit ESD of A).

One may show, for any μ_A with bounded support, that

$$m(z) = m_A(z + m(z))$$

has a unique root $m(z) \in \mathbb{C}^+$ when $z \in \mathbb{C}^+$. Then $m_n(z) \to m(z)$ a.s.

Thm (Pastur '72): The equation $m(z) = m_A(z + m(z))$ has a unique root $m(z) \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$, which defines the Stieltjes transform of a measure μ on $\mathbb{R}$. For $X \sim GOE(n)$ and A deterministic, symmetric, if the ESD of A converges weakly to μ_A w/ Stieltjes transform $m_A(z)$ as $n\to\infty$, then that of $X+A$ converges weakly a.s. to μ.

Writing μ_{sc} as the semi-circle law on $[-2,2]$, we write

$$\mu = \mu_{sc} \boxplus \mu_A,$$ the <u>additive free convolution</u> of μ_{sc} and μ_A.

<u>Lindeberg exchange method</u>

We illustrate the idea by proving the following quantitative CLT:

Thm: Let $X_1,\dots,X_n$ be iid random variables with $\mathbb{E}[X_i]=0$, $\mathbb{E}[X_i^2]=1$, $\mathbb{E}[|X_i|^3]=m_3$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be three-times differentiable, with uniformly bounded third derivative. Then for $\xi\sim\mathcal{N}(0,1)$,

$$\left|\mathbb{E}\left[\varphi\left(\frac{X_1+\dots+X_n}{\sqrt{n}}\right)\right]-\mathbb{E}[\varphi(\xi)]\right| \le \frac{m_3+\sqrt{\frac{8}{\pi}}}{6\sqrt{n}}\cdot \sup_{x\in\mathbb{R}}|\varphi'''(x)|.$$

Proof: Write $\xi = \frac{Z_1+\dots+Z_n}{\sqrt{n}}$, where $Z_i \overset{iid}{\sim} \mathcal{N}(0,1)$. We "swap" X_i one by one for Z_i. Let

$$S^{(i)} = \frac{X_1+\dots+X_i+Z_{i+1}+\dots+Z_n}{\sqrt{n}}.$$

Then we bound $$\left|\mathbb{E}[\varphi(S^{(i)})]-\mathbb{E}[\varphi(S^{(i-1)})]\right|$$

for $i=n,n-1,\dots,1$, and then sum over i.

Let $T^{(i)} = \frac{X_1+\dots+X_{i-1}+Z_{i+1}+\dots+Z_n}{\sqrt{n}}$. Then

- $T^{(i)}$ is independent of both X_i and Z_i.
- $S^{(i)}=T^{(i)}+\frac{X_i}{\sqrt{n}}$, $S^{(i-1)}=T^{(i)}+\frac{Z_i}{\sqrt{n}}$.

So $$\varphi(S^{(i)}) = \varphi(T^{(i)})+\varphi'(T^{(i)})\cdot\frac{X_i}{\sqrt{n}}+\frac{\varphi''(T^{(i)})}{2}\cdot\frac{X_i^2}{n}+\frac{\varphi'''(Y_i)}{6}\cdot\frac{X_i^3}{n^{3/2}},$$

$$\varphi(S^{(i-1)}) = \varphi(T^{(i)})+\varphi'(T^{(i)})\cdot\frac{Z_i}{\sqrt{n}}+\frac{\varphi''(T^{(i)})}{2}\cdot\frac{Z_i^2}{n}+\frac{\varphi'''(W_i)}{6}\cdot\frac{Z_i^3}{n^{3/2}}$$

for some (random) $Y_i, W_i\in\mathbb{R}$.

Since $T^{(i)}$ is independent of X_i, Z_i, and the first two moments of X_i and Z_i match,

$$\left|\mathbb{E}[\varphi(S^{(i)})]-\mathbb{E}[\varphi(S^{(i-1)})]\right| = \left|\mathbb{E}\left[\frac{\varphi'''(Y_i)}{6}\cdot\frac{X_i^3}{n^{3/2}}-\frac{\varphi'''(W_i)}{6}\cdot\frac{Z_i^3}{n^{3/2}}\right]\right|$$

$$\le \frac{\sup_x|\varphi'''(x)|}{6n^{3/2}}\cdot\underbrace{\mathbb{E}[|X_i|^3+|Z_i|^3]}_{=m_3+\sqrt{\frac{8}{\pi}}}.$$

Summing over $i=1,\dots,n$ gives the result.

We now apply this to the Stieltjes transform of a random matrix:

Lemma: Let $X,\tilde{X}\in\mathbb{R}^{n\times n}$ be two Wigner matrices such that

- $\mathbb{E}[X_{ij}]=\mathbb{E}[\tilde{X}_{ij}]=0$ for all i,j
- $\mathbb{E}[X_{ij}^2]=\mathbb{E}[\tilde{X}_{ij}^2]=\frac{1}{n}$ for all $i\neq j$
- $\mathbb{E}[|X_{ij}|^3]$, $\mathbb{E}[|\tilde{X}_{ij}|^3]\le\frac{C}{n^{3/2}}$ for all i,j and

  $\mathbb{E}[|X_{ii}|^2]$, $\mathbb{E}[|\tilde{X}_{ii}|^2]\le\frac{C}{n}$ for all i.

Let $m_n(z)$ and $\tilde{m}_n(z)$ be the Stieltjes transforms of the ESDs of $X+A$ and $\tilde{X}+A$, for any deterministic, symmetric $A\in\mathbb{R}^{n\times n}$. Then

$$\left|\mathbb{E}[m_n(z)]-\mathbb{E}[\tilde{m}_n(z)]\right|\le C'\left(\frac{1}{\sqrt{n}(\operatorname{Im} z)^4}+\frac{1}{n(\operatorname{Im} z)^3}\right)$$

for all $z\in\mathbb{C}^+$.

Proof: We swap the entries $(X_{ij})_{i\le j}$ for $(\tilde{X}_{ij})_{i\le j}$, one at a time.

First do the off-diagonal: Order the pairs (i,j) with $i<j$ arbitrarily.

Define $Y^{(ij)} \in \mathbb{R}^{n\times n}$ as the matrix where

- $Y^{(ij)}_{kl} = X_{kl}$ and $Y^{(ij)}_{lk} = X_{lk}$ for (k,l) preceding (i,j).
- $Y^{(ij)}_{kl} = \tilde{X}_{kl}$ and $Y^{(ij)}_{lk} = \tilde{X}_{lk}$ for (k,l) following (i,j).
- $Y^{(ij)}_{ij} = 0$ and $Y^{(ij)}_{ji} = 0$.

Let $X^{(ij)} = Y^{(ij)} + X_{ij}(\Delta_{ij}+\Delta_{ji})$

$\tilde{X}^{(ij)} = Y^{(ij)} + \tilde{X}_{ij}(\Delta_{ij}+\Delta_{ji})$.

So $X^{(ij)} \mapsto \tilde{X}^{(ij)}$ swaps the (i,j) and (j,i) entries of X for $\tilde{X}$.

Let $m_n^{(ij)} = \frac{1}{n}\mathrm{Tr}\,(X^{(ij)}+A-zI)^{-1} \equiv \frac{1}{n}\mathrm{Tr}\,G_n^{(ij)}$

$\tilde{m}_n^{(ij)} = \frac{1}{n}\mathrm{Tr}\,(\tilde{X}^{(ij)}+A-zI)^{-1} \equiv \frac{1}{n}\mathrm{Tr}\,\tilde{G}_n^{(ij)}$

$h_n^{(ij)} = \frac{1}{n}\mathrm{Tr}\,(Y^{(ij)}+A-zI)^{-1} \equiv \frac{1}{n}\mathrm{Tr}\,H_n^{(ij)}$.

Key identity: $A^{-1}-B^{-1} = A^{-1}(B-A)B^{-1}$.

Repeatedly applying this,

$$
\begin{aligned}
A^{-1}-B^{-1} &= A^{-1}(B-A)B^{-1} \\
&= A^{-1}(B-A)A^{-1} - A^{-1}(B-A)(A^{-1}-B^{-1}) \\
&= A^{-1}(B-A)A^{-1} - A^{-1}(B-A)A^{-1}(B-A)B^{-1} \\
&= \dots = \sum_{k=1}^{\ell} (-1)^{k-1}\left(A^{-1}(B-A)\right)^k B^{-1} \text{ for any } \ell\ge 1.
\end{aligned}
$$

In particular: Since $(X^{(ij)}+A-zI)-(Y^{(ij)}+A-zI) = X_{ij}(\Delta_{ij}+\Delta_{ji}) \equiv \Delta$,

$$
\begin{aligned}
h_n^{(ij)} - m_n^{(ij)} &= \frac{1}{n}\mathrm{Tr}\left[H_n^{(ij)} - G_n^{(ij)}\right] \\
&= \frac{1}{n}\mathrm{Tr}\Big[H_n^{(ij)}\Delta H_n^{(ij)} - H_n^{(ij)}\Delta H_n^{(ij)}\Delta H_n^{(ij)} \\
&\qquad + H_n^{(ij)}\Delta H_n^{(ij)}\Delta H_n^{(ij)}\Delta G_n^{(ij)}\Big]
\end{aligned}
$$

Similarly letting $\tilde{\Delta} = \tilde{X}_{ij}(\Delta_{ij}+\Delta_{ji})$,

$$
\begin{aligned}
h_n^{(ij)} - \tilde{m}_n^{(ij)} &= \frac{1}{n}\mathrm{Tr}\left[H_n^{(ij)} - \tilde{G}_n^{(ij)}\right] \\
&= \frac{1}{n}\mathrm{Tr}\Big[H_n^{(ij)}\tilde{\Delta} H_n^{(ij)} - H_n^{(ij)}\tilde{\Delta} H_n^{(ij)}\tilde{\Delta} H_n^{(ij)} \\
&\qquad + H_n^{(ij)}\tilde{\Delta} H_n^{(ij)}\tilde{\Delta} H_n^{(ij)}\tilde{\Delta}\tilde{G}_n^{(ij)}\Big].
\end{aligned}
$$

Use now: $H_n^{(ij)}$ is independent of both X_{ij} and $\tilde{X}_{ij}$, and

X_{ij} and $\tilde{X}_{ij}$ have matching first and second moments.

Then $\left|\mathbb{E}\left[m_n^{(ij)} - \tilde{m}_n^{(ij)}\right]\right|$

$$
\begin{aligned}
&\le \underbrace{\left|\mathbb{E}\left[\frac{1}{n}\mathrm{Tr}\, H_n^{(ij)}\tilde{\Delta} H_n^{(ij)}\tilde{\Delta} H_n^{(ij)}\tilde{\Delta}\tilde{G}_n^{(ij)}\right]\right|}_{\mathrm{II}} \\
&\quad + \underbrace{\left|\mathbb{E}\left[\frac{1}{n}\mathrm{Tr}\, H_n^{(ij)}\Delta H_n^{(ij)}\Delta H_n^{(ij)}\Delta G_n^{(ij)}\right]\right|}_{\mathrm{I}}.
\end{aligned}
$$

Note that

$$\mathrm{I} = \left|\mathbb{E}\left[\frac{X_{ij}^3}{n}\cdot\sum_{k=1}^{n}\left(H(\Delta_{ij}+\Delta_{ji})H(\Delta_{ij}+\Delta_{ji})H(\Delta_{ij}+\Delta_{ji})G\right)_{kk}\right]\right|$$

and $\sum_{k=1}^{n}(H\Delta_{ij}H\Delta_{ij}H\Delta_{ij}G)_{kk} = \sum_{k=1}^{n} H_{ki}H_{ji}H_{ji}G_{jk} = (GH)_{ji}H_{ji}H_{ji}$.

Since $\|H\|\le\frac{1}{\mathrm{Im}\,z}$, $\|G\|\le\frac{1}{\mathrm{Im}\,z}$, this is bounded by $\|GH\|\cdot\|H\|\cdot\|H\| \le \frac{1}{(\mathrm{Im}\,z)^4}$.

Then $\mathrm{I} \le \frac{1}{n(\mathrm{Im}\,z)^4}\cdot\mathbb{E}\left[|X_{ij}|^3\right] \le \frac{C}{n^{5/2}(\mathrm{Im}\,z)^4}$. Similarly for II.

There are $\binom{n}{2}$ such swaps, so summing over $\binom{i}{j}$ shows

$$\left|\mathbb{E}[m_n(z)] - \mathbb{E}[\check{m}_n(z)]\right| \le \frac{C}{\sqrt{n}\,(\operatorname{Im} z)^4}$$

where $\check{m}_n(z) = \frac{1}{n}\operatorname{Tr}(\check{X}+A-zI)^{-1}$ and $\check{X}$ has off-diag entries equal to $\tilde{X}$, diag entries equal to X.

Now do the same for the diagonal: Define $Y^{(i)} \in \mathbb{R}^{n\times n}$ s.t.

- $Y^{(i)}_{jj} = X_{jj}$ for $j<i$
- $Y^{(i)}_{jj} = \tilde{X}_{jj}$ for $j>i$
- $Y^{(i)}_{ii} = 0$
- $Y^{(i)}_{jk} = \tilde{X}_{jk}$ for $j \ne k$.

Set $X^{(i)} = Y^{(i)} + \overbrace{X_{ii}\Delta_{ii}}^{\Delta}$, $\tilde{X}^{(i)} = Y^{(i)} + \overbrace{\tilde{X}_{ii}\Delta_{ii}}^{\tilde{\Delta}}$,

define analogously $m_n^{(i)}, \tilde{m}_n^{(i)}, h_n^{(i)}$. Then, since first moments match,

$$\left|\mathbb{E}m_n^{(i)} - \mathbb{E}\tilde{m}_n^{(i)}\right| \le \left|\mathbb{E}\left[\frac{1}{n}\operatorname{Tr} H_n^{(i)}\Delta H_n^{(i)}\Delta G_n^{(i)}\right]\right| + \left|\mathbb{E}\left[\frac{1}{n}\operatorname{Tr} H_n^{(i)}\tilde{\Delta} H_n^{(i)}\tilde{\Delta}\tilde{G}_n^{(i)}\right]\right|$$

$$\le \frac{1}{n(\operatorname{Im} z)^3}\left(\mathbb{E}[|X_{ii}|^2] + \mathbb{E}[|\tilde{X}_{ii}|^2]\right) \le \frac{C}{n^2(\operatorname{Im} z)^3}.$$

Number of diagonal swaps is n, so summing over i,

$$\left|\mathbb{E}[\check{m}_n(z)] - \mathbb{E}[\tilde{m}_n(z)]\right| \le \frac{C}{n(\operatorname{Im} z)^3}.$$

Combining yields the result.

Proof #3 of semi-circle law, when X satisfies LSI:

By this result, $\mathbb{E}[m_n(z)]$ is the same for X as for the GOE.

Then the integration-by-parts argument shows $\mathbb{E}[m_n(z)] \to m(z)$.

Concentration of $m_n(z)$ follows from the LSI, since $x \mapsto \frac{1}{x-z}$ is $\frac{1}{(\operatorname{Im} z)^2}$-Lipschitz.

So $m_n(z) \to m(z)$ a.s.

Problem 3: Recall that if $A \in \mathbb{R}^{n\times n}$ has ESD converging to μ_A, with Stieltjes transform $m_A(z) = \int \frac{1}{x-z}\,d\mu_A(x)$, then the ESD of $X+A$ for Wigner X converges to μ w/ Stieltjes transform satisfying

$$m(z) = m_A(z + m(z)) \qquad (*).$$

(a) For fixed $z \in \mathbb{C}^+$, devise a numerical algorithm to solve $(*)$ for $m(z) \in \mathbb{C}^+$, and implement this algorithm.

(b) Write a program to plot an approximation for the probability density function of μ. How close is the ESD of $X+A$ to this density for $n=100$? 500? 2500?

(c) Derive the analogue of $(*)$ for the matrix $\sigma X + A$, for arbitrary scaling $\sigma > 0$, and let the limit ESD be μ_σ.

For fixed A (numerically), how does the density of μ_σ change as σ increases from 0 to ∞? What happens to the number of intervals of support of μ_σ?

Explore this in some settings where μ_A has multiple disjoint intervals of support.