Spectral norm

Let $X \in \mathbb{R}^{m \times n}$ be a Wigner matrix. Its ESP, $\mu$, is the semicircle law on $[-2,2]$.

The statement $\mu \Rightarrow \mu$ weakly means $\mu_n \Rightarrow \mu$ (in $[a,b]$) for any fixed interval $[a,b]$, i.e.,

$$\int \lambda : \lambda \in [a,b] \Rightarrow \mu_n(\ell, \lambda) \approx \mu(\ell, \lambda) + o(n).$$

It does not address the following questions:

4) For a shrinking interval $[a_n,b_n]$ with $b_n-a_n = o(n)$, do we have

$$\int \lambda : \lambda \in [a_n,b_n] \Rightarrow \mu_n(\ell, \lambda) \approx \mu(\ell, \lambda) + o(n)?$$

5) For an interval $[a,b]$ outside the support $[2,2]$ of $\mu$, are there in fact no eigenvalues of $X$ in $[a,b]$?

(Wake convergence $\mu \Rightarrow \mu$ only implies the number of eigenvalues is $o(n)$.)

Check via simulation: If $X \sim N(0,1)$, when $E[\ell] = 0$, $E[\ell^2] = 1$ but $E[\ell^4] = o(1)$, then we get:

$\mu_n$ weakly but $\|X\|_2 \Rightarrow 2$.

The Stieltjes transform method can address both (4) and (5). The moment method can address (5).

Moment method idea: For even integers $k$,

$$\|X\|_k^\ell = \max |\lambda|^k \leq \sum |\lambda|^k = T \cdot X^k \approx n^{\ell} \|X\|_k^\ell.$$

So $\|X\|_k^\ell \cdot (T \cdot X^k)^\ell \approx n^{\ell} \|X\|_k^\ell$. If $k = 2m + 1$, then $n^{\ell \cdot m} \approx n^{\ell}$ and

$$\|X\|_k^\ell \cdot \|X\|_k^\ell \approx n^{\ell \cdot m} \|X\|_k^\ell$$

is an upper bound for $\|X\|_k^\ell$ that is tight up to a small multiplicative factor for large $C$. (Farrell, Karpel '80)

Then: Suppose $X$ is a Wigner matrix, $X \sim N(0,1)$, where $E[\ell^2] = \ell \cdot k$ for some constant $k > 0$ and any $\ell > 0$. Then $\|X\|_2 \Rightarrow 2$ a.s.

Proof: Since $\mu \Rightarrow \mu$, this implies $X$ has $\ell \cdot m$ eigenvalues in any interval $[-\lambda, \lambda]$. So $\max_{\lambda > \lambda} \|X\|_2 \Rightarrow 2$ a.s., since $\sum_{\lambda > \lambda} \max_{\lambda > \lambda} \|X\|_2 \Rightarrow 2$ a.s. For the upper bound: Fix $\ell > 0$. Suppose we can show

$$\|X\|_2^\ell \leq (2 \cdot k)^{\ell}$$

for some even $k > 2$ and all large $n$,

where $C \cdot \log \frac{2 \cdot k}{k \cdot \ell} \gg 2$. Then

$$\|X\|_2^\ell \leq (2 \cdot k)^{\ell}$$

This yields $\max \|X\|_2 \Rightarrow 2$ a.s. Since $\ell > 0$ is arbitrary, it remains to show (3). Recall

$$\text{IE}[\ell^k] = \text{IE} \left[ \sum_{\ell} \ell^k \cdot X_{\ell} \cdot X_{\ell+1} \cdot \ldots \cdot X_{\ell+k} \right]$$

$$= \sum_{\ell} \text{IE} [X_{\ell} \cdot X_{\ell+1} \cdot \ldots \cdot X_{\ell+k}]$$

where $(\ell, \ell+1, \ldots, \ell+k)$ is a path of $\ell$, $\ell+1, \ldots, \ell+k$ on graph with $n$ vertices.
For a path \( \gamma \), let \( \ell(\gamma) \): \# distinct visited vertices

\( e(\gamma) \): \# distinct traversed (undirected) edges

Recall \( v(\gamma) = e(\gamma) + 1 \), since \( \gamma \) walks over a connected graph.

First bound \( \frac{1}{n^2} E[\ell(X_{i:n} \cdot X_{i:n})] \):

- If some edge is traversed only once, then some \( X_i \) appears only once in this product, so this is 0.

- More generally, let \( k_1, \ldots, k_n \) be the number of times the distinct edges are traversed, and suppose each \( k_i \geq 2 \). Then

\[
E[\ell(X_{i:n} \cdot X_{i:n})] = \frac{1}{n^2} \sum \frac{1}{n!} \sum \prod_{i=1}^{n} k_i \prod_{i=1}^{n} k_{i+} \prod_{i=1}^{n} k_{i-}
\]

\[
\leq \frac{1}{n^2} \sum \frac{1}{n!} \sum \prod_{i=1}^{n} k_i \prod_{i=1}^{n} k_{i+} \prod_{i=1}^{n} k_{i-} = \frac{1}{n^{n+1}} \frac{1}{n!} \prod_{i=1}^{n} k_i
\]

Actually, we'll need a tighter bound. If edge \( i \) is not a self-loop and \( k_i \geq 2 \), then we may replace \( k_i k_{i-} \) by \( 1 \). So in fact

\[
E[\ell(X_{i:n} \cdot X_{i:n})] \leq \frac{1}{n^2} \sum \frac{1}{n!} \sum \prod_{i=1}^{n} k_i \prod_{i=1}^{n} k_{i+} \prod_{i=1}^{n} k_{i-} = \frac{1}{n^{n+1}} \frac{1}{n!} \prod_{i=1}^{n} k_i
\]

Let \( p = \sum_{i=1}^{n} k_i \) if edge \( i \) is self-loop

\( q = \sum_{i=1}^{n} k_i \) if edge \( i \) is not self-loop

\[
E[\ell(X_{i:n} \cdot X_{i:n})] \leq \frac{1}{n^{n+1}} \frac{1}{n!} \prod_{i=1}^{n} k_i
\]

We'll want an upper bound on \( p q \) in terms of \( \frac{1}{2} (v(\gamma) + 1) \).

Two observations:

1. Let \( v(\gamma) \) be number of distinct non-self-loop edges. Then

\[
v(\gamma) \leq \frac{e(\gamma)}{2} + \frac{k - p q}{2}
\]

2. By a pigeonhole argument, if each edge is traversed at least twice

\[
e(\gamma) \leq \frac{k}{2} - \frac{p q}{2} \leq \frac{k}{2} - \frac{k - p q}{2}
\]

Then \( p q \leq 6 (\frac{k}{2} - e(\gamma)) \leq 6 (\frac{k}{2} - v(\gamma)) \).

We get

\[
E[\ell(X_{i:n} \cdot X_{i:n})] \leq \frac{1}{n^{n+1}} \frac{1}{n!} \prod_{i=1}^{n} (\frac{k}{2} - e(\gamma))
\]

Second, count paths \( \gamma \) that visit \( v(\gamma) \) vertices, and traverse each edge at least twice:

- Corresponding to each path is a "canonical path" that is equivalent up to a permutation of the labels \( 1, \ldots, n \) and order of distinct vertices visited are \( 1, 2, \ldots, \frac{1}{2} (v(\gamma) + 1) \) in visited order.

Each canonical path is equivalent to

- Exactly \( n(n-1)(n-2) \cdots (n-v(\gamma)+1) \) \( n \)-paths.

- For a canonical path, divide the steps into three cases:

  - \( + \): This step visits a new vertex not previously visited,

  - \( - \): This step is the second traversal (in either direction) of an edge corresponding to a previous +

  - \( * \): Everything else (this case includes all self-loops).
We may describe a canonical path by first describing the sequence of +, −, and * steps:

+ + + + ∗ ∗ ∗ ∗ ∗

Then providing vertex labels compatible with the above steps:

2 3 4 3 5 2 1 2 3 5 2 1.

Note: The sequence +, −, and ∗ does not uniquely specify a sequence of compatible vertex labels. However, we have the following observations:

- Each + does have a unique vertex label, but the path is canonical. (These must be 2, 3, 4, v(k), in order.)
- These steps + form a tree on two vertices 1, v(k).
- Before the first ∗, all labels are uniquely specified.
- If, in addition, we specify the label of each ∗ and the step preceding each ∗ (i.e., what are the two vertices that the ∗ step jumped between), then all remaining labels are uniquely specified.

This is because each chain of consecutive −s is a connected path of distinct edges on the tree. If we specify the start and end vertices of this path, then the path is unique.

So given a sequence of +, −, and ∗, the number of labelings is at most \( \sqrt{k} \cdot \frac{k}{\sqrt{n^2 + k^2}} \leq \sqrt{n} \cdot \frac{k}{\sqrt{n^2 + k^2}} \).

There are exactly \( \sqrt{k} \cdot \frac{k}{\sqrt{n^2 + k^2}} \) + steps, so also \( \sqrt{k} \cdot \frac{k}{\sqrt{n^2 + k^2}} \) − steps. Then

\[
\text{Number of sequences of +, −, and ∗ is } \leq \left( \frac{k^2}{n^2 + k^2} \right)^{k} \leq \frac{k^2}{n^2 + k^2}.
\]

Total number of paths visiting \( v(k) \) vertices is \( \leq n \cdot \frac{k}{\sqrt{n^2 + k^2}} \).

Combining these, and noting \( 1 \leq \sqrt{k} \leq \frac{k}{\sqrt{n^2 + k^2}} \) for paths visiting each edge at least twice,

\[
\mathbb{E}[T_{v(k)}] = \sum_{v=1}^{n} \sum_{x=1}^{n} \mathbb{E}[X_{v,x}] = \sum_{v=1}^{n} \frac{n^2}{2} \cdot \frac{1}{k} \cdot \frac{k}{\sqrt{n^2 + k^2}} \frac{k}{n} \cdot \frac{k}{n}.
\]

\[
\leq n \cdot \frac{k}{\sqrt{n^2 + k^2}} \cdot \left( \frac{k}{\sqrt{n^2 + k^2}} \right)^{k} \leq n \cdot \frac{k}{\sqrt{n^2 + k^2}} \cdot \left( \frac{k}{\sqrt{n^2 + k^2}} \right)^{k} \leq n \cdot \frac{k}{\sqrt{n^2 + k^2}} \cdot \left( \frac{k}{\sqrt{n^2 + k^2}} \right)^{k}.
\]

For any \( k > C \ln n \) and \( n^2 \ln(C) \) large enough.

Note: \( n^2 \ln(C) = \ln n + \ln(k) \ln(n) \) by \( 2 \leq e^{k \cdot \ln(\ln(n))} = (\ln(n))^{k} \).

For \( k > C \ln n \) and \( C \) large enough constant, this shows \( (\sqrt{k}) \) as desired.
Overview of local semicircle law

Notation: We write \( X(x) \sim \Psi(x) \) if, for any \( \varepsilon, D > 0 \), there is \( n(\varepsilon, D) \geq 0 \)
s.t. \( \mathbb{P}( |X(x)| > n^\varepsilon \Psi(x) ) \leq n^{-D} \) for all \( n \geq n(\varepsilon, D) \).

"For all large \( n \), \( |X(x)| \) is at much larger than \( \Psi(x) \) with high probability."

We also write this as \( X(x) \sim \Omega(\Psi(x)) \).

We'll say this holds uniformly over \( x \in D \) if

\[
\mathbb{P}( \exists \varepsilon > 0 : |X(x)| > n^\varepsilon \Psi(x) ) \leq n^{-D} \text{ for all } n \geq n(\varepsilon, D).
\]

Then (Local semicircle law): Let \( X \in \mathbb{R}^{n \times n} \) be a Wigner matrix s.t.
\( \mathbb{E}[|X_{ij}|^2] \leq C \) for some constants \( C > 0 \) and all \( k > 0 \).

Fix any constants \( C, \varepsilon > 0 \) and set

\[
D = \{ x \in \mathbb{C} : \text{Re } x \in [-C, C], \text{Im } x \in [-\varepsilon, \varepsilon] \}.
\]

Let \( G_0(x) = (x-I)^{-1} \) and \( m(x) = \text{tr}[G_0(x)] \).\] Then

(a) Uniformly over \( x \in D \), \( |m(x) - m(0)| \lesssim \frac{1}{n} \).

(b) Uniformly over \( \{ x \in D : \text{Re } x \in [-2; 2] \} \)

\[
|\text{Im } m(x) - m(0)| \lesssim \frac{1}{n(\varepsilon)} + \frac{1}{n^\varepsilon}.
\]

(c) Uniformly over \( x \in D \), for all \( i \in \{ 1, \ldots, n \} \),

\[
|G(x)_{ii} - m(x)| \lesssim \sqrt{\frac{\text{Im } m(x) + 1}{n}}.
\]

Proof: This is a (non-trivial) generalization of the Schur complement argument from Lecture 3. See Erdos, Kratzers, Yau, Yin, 2013

"Local Semicircle Law for a General Class of Random Matrices."

A full exposition is also given in Benaych-Georges, Kratzers, Yau, 2018

"Lectures on the local semicircle law for Wigner matrices."

Remarks: (a) and (b) give bounds on \( m(x) - m(0) \) for \( x \in D \) approximately the real line at a rate \( n^{-1} \). We'll see that simple approximation of \( m(x) \) by \( m(0) \) over intervals \( [a, b] \) with \( |a - b| = n^{-1/2} \).

(c) gives an entrywise bound on the resolvent, and shows

\[
G(x) \approx m(x) I \text{ in an entrywise sense. We'll see that this is useful for studying eigenvalues of } X.
\]

Note that the density of the semicircle law is bounded, hence so is \( \text{Im } m(x) \) over \( x \in D \). So (c) implies the simpler bound

\[
|G(x)_{ii} - m(x)| \lesssim \frac{1}{n^{1/2}}.
\]

We get more refined bounds by using (see Problem 3)

\[
\text{Im } m(x) \approx \left\{ \begin{array}{ll}
\frac{\text{Im } x}{n^{1/2}} & \text{if } \text{Re } x \in (-2; 2) \\
0 & \text{otherwise}
\end{array} \right.,
\]

Proposition: Suppose, at some \( x \), that \( \text{Im } m(x) \approx \text{Im } m(0) < 1 \). Then

\( m(x) \) has no eigenvalues in the interval \( [1/n, 1] \).
**Proof:** We have
\[
\text{Im } m_n(\pm i\eta) = \text{Im } \frac{1}{\eta} \sum_{\ell=1}^{\infty} \frac{(\pm i\eta)^{2\ell-1}}{(\ell,\ell)} \cdot \eta^{2\ell-1} \cdot \xi_{\ell-1}^*(x)
\]
\[
\geq \frac{1}{2\eta} \sum_{\ell=1}^{\infty} \frac{(\pm i\eta)^{2\ell-1}}{(\ell,\ell)} \cdot \eta^{2\ell-1} \cdot \xi_{\ell-1}^*(x)
\]
\[
\geq \frac{1}{2\eta} \sum_{\ell=1}^{\infty} \frac{(\pm i\eta)^{2\ell-1}}{(\ell,\ell)} \cdot \eta^{2\ell-1} \cdot \xi_{\ell-1}^*(x)
\]
Thus \( \text{Im } m_n(\pm i\eta) \geq \) eigenvalues in \( [2^{-n} C, 0] \).

If this is \( \leq 0 \), then the eigenvalues must be \( 0 \).

**Cor:** For any constants \( C, \xi > 0 \), \( X \) has no eigenvalues in \( [2^{-n} C, 0] \) at all large \( n \).

**Proof:** Set \( \eta = n^{-2\beta} \).

Consider
\[
L = \{ x \in \mathbb{R}^d : \text{Im } \eta \leq x \leq 2\eta, x \in [2^{n-1} C, 0] \}
\]

Uniformly over \( x \in L \):
\[
\text{Im } m(x) \leq \frac{\eta}{\eta + n^{-2\beta}} \leq \frac{\eta}{\eta \cdot n^{-2\beta}} = \frac{n^{-2\beta}}{\eta}
\]
\[
\text{Im } m(x) \cdot m(x) \leq \frac{1}{n^{2\beta}} \cdot \frac{1}{n^{2\beta}} = \frac{1}{n^{4\beta}}
\]

Then \( \text{Im } m(x) \cdot m(x) \leq \frac{1}{n^{4\beta}} \leq \frac{1}{n^{4\beta}} \)

for all large \( n \). On this event, \( \text{Im } m(x) \geq 2^{-n^{2\beta} + 2} \)

for all \( x \in L \), so there are no eigenvalues in \( [2^{-n} C, 0] \).

Combining with a bound \( \|X\|\leq C \) a.s., we get \( \|X\| \leq 2 \) a.s.

**Cor:** For any constants \( C, \xi > 0 \), \( X \) has no eigenvalues in \( [2^{-n} C, 0] \) a.s., for all large \( n \).

**Proof:** Consider
\[
L = \{ x \in \mathbb{R}^d : \text{Im } \eta = 2^{-n^{2\beta} + 2}, x \in [2^{-n^{2\beta} + 2} C, 0] \}
\]

Uniformly over \( x \in L \):
\[
\text{Im } m(x) \leq \frac{\eta}{\eta + n^{-2\beta}} \leq \frac{\eta}{\eta \cdot n^{-2\beta}} = \frac{n^{-2\beta}}{\eta}
\]
\[
\text{Im } m(x) \cdot m(x) \leq \frac{1}{n^{2\beta}} \cdot \frac{1}{n^{2\beta}} = \frac{1}{n^{4\beta}}
\]

So also \( \text{Im } m(x) \leq 0 \) a.s., for all \( x \in L \).

**Proposition:** (a) (Cauchy–integral) Suppose \( \mu \) has bounded support and

strictly non-trivial \( m(\cdot) \). Let \( E : \mathbb{R}^d \to \mathbb{R}^d \) and suppose \( E \) extends analytically to a domain USC containing \( \text{supp} \mu \). Then for \( \mu \) (counterclockwise) contour \( \gamma \), \( \mu \) endously supports
\[
\int f(\cdot) d\mu(\cdot) \neq 0.
\]

(b) (Helffer–Jörres) Suppose \( E : \mathbb{R}^d \to \mathbb{R}^d \) is \( (\text{dim}) \)-times continuously differentiable, with bounded support. Define an "almost analytic" extension by
\[
g(x+y) = f(y) + \xi f(x+y) e^{\xi x f(x) + \xi y f(y) + f(x+y)},
\]

where \( f(x) \) is smooth, \( \equiv 1 \) in a neighborhood of \( 0 \), symmetric around \( 0 \),

compact supports.

Then:
\[
\int f(x+y) d\mu(\cdot) = \frac{1}{2\pi} \int f(x+y) e^{i(x+y)^T \xi} \cdot e^{i(x+y)^T \xi} \cdot m(x+y) \cdot dx
\]
Proof: For (a), Cauchy integral formula gives
\[ \ell(x) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(t)}{t-x} \, dt \]
for any \( \gamma \) in the contour. Then
\[ \int \ell(\gamma) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(t) - f(\gamma)}{t-x} \, dt = \frac{-1}{2\pi i} \oint_{\gamma} \frac{f(t) - f(\gamma)}{t-x} \, dt. \]

For (b), an application of Green's Theorem gives
\[ \ell(x) = \frac{1}{2\pi} \int_{\gamma} (\partial_x \gamma \cdot \gamma(x, \gamma))^2 \, dt, \]
See [AG, Proof of Lemma 5.5]. Then (c) follows similarly.

Cor: Uniquely over (\(d\) independent) intervals \( I \in [G, G] \),
\[ \left| \frac{1}{n} \ell(x) \right| \leq \mu(I) \leq n^{-1} \]
where \( \mu \) is the semicircle law.

Proof: Let \( \mu_n \) be the ESP of \( X \). Fix \( \eta = n^{-1/2} \). Define \( E : \mathbb{R} \to [0, \infty] \) by
\[ f(x) < 1 \text{ if } x \in I \]
\[ f(x) = 0 \text{ if } \text{dist}(x, I) \geq \eta \]
\[ |f'(x)| < \frac{C}{\eta}, \quad |f''(x)| < \frac{C}{\eta^2}. \]
Define \( X : \mathbb{R} \to [0, \infty] \) by \( X(x) = 1 \) for \( x \in [E^{-1}(1), E^{-1}(2)] \), \( X(x) = 0 \) for \( x \notin [0, 2] \), \( |X(x)| \leq C \) for all \( x \). Apply Helly's theorem with \( b \):
\[ \int (\ell(x) f(x) - \ell(x_f) f(x) - \ell(x_f) f(x_f)) \, dx. \]
\[ \ell(x) f(x) - \ell(x_f) f(x_f) = \frac{1}{2\pi i} \int_{\gamma} (f(t) - f(\gamma)) (x_f - \gamma(x, \gamma)) \, dt. \]

Note: \( \ell(x) f(x) = -i \bar{y} \ell \bar{y} \ell(x). \) Then
\[ (\ell, \ell) (X(x) f(x), \ell(x) f(x)) = i \bar{y} X(x) \ell \bar{y} \ell(x) + X(x) f(x) \ell(x) \ell(x). \]

We have \( X(x) = 0 \) unless \( y \in [E 1, E 2] \). For such \( y \),
\[ \ell(x) f(x) - \ell(x_f) f(x_f) \leq \frac{1}{2\pi i} \int_{\gamma} (f(t) - f(\gamma)) (x_f - \gamma(x, \gamma)) \, dt. \]
\[ \leq \frac{1}{n} \int_{\eta} (f(t) + f(\gamma)) \, dt < \frac{1}{n}. \]
For the first term, consider first
\[ I = \int_{\eta} \int_{e^1} \int_{e^2} \int_{e^3} \int_{e^4} \int_{e^5} \int_{e^6} (m(x, y) - m(x+iy)) \, dy. \]
Integrate by parts to find \( x \), then as \( x \)
\[ \int_{\eta} \int_{e^1} \int_{e^2} \int_{e^3} \int_{e^4} \int_{e^5} \int_{e^6} (m(x, y) - m(x+iy)) \, dy. \]

So \( |I| \leq \frac{1}{n}. \)
Consider next
\[ \int_{\eta} \int_{e^1} \int_{e^2} \int_{e^3} \int_{e^4} \int_{e^5} \int_{e^6} (m(x, y) - m(x+iy)) \, dy. \]
Then \( \text{Re} II = \sum_{x=1}^{n} \sum_{y=1}^{n} x(y) \text{Im}(\text{Im}(x)-\text{Im}(y)) \text{d}x \text{d}y \).

Apply \( D \) so \( \text{Im}(x) = \sum_{y=1}^{n} x(y) \), and similar for \( y \).

Then

\[
|\text{Re} II| \leq \sum_{x=1}^{n} \sum_{y=1}^{n} \left( (x(y) \text{Im}(x(y)) + y(y) \text{Im}(y(y))) \text{d}x \text{d}y \right)
\]

\(所以, \sum_{x=1}^{n} \sum_{y=1}^{n} x(y) \text{Im}(x(y)) \leq n^{-1/2}. \)

We also have \( |\text{Im} \mu - \text{Re} \mu| = |\text{Re} (\text{Im} \mu - \text{Re} \mu)| \leq n^{-1/2}. \)

Here \( \eta \geq 0 \) is arbitrary, so \( |\text{Im} \mu - \text{Re} \mu| \leq n^{-1/2}. \)

Finally, define \( E^* E_1 = \frac{1}{n} \sum_{x=1}^{n} \sum_{y=1}^{n} x(y) \text{Im}(x(y)) \theta_2 \).

Applying the above to \( I \) and \( E \), and \( \sum_{x=1}^{n} \sum_{y=1}^{n} x(y) \text{Im}(x(y)) \), we yield the result.

Cor: Let \( \mu \), \( \nu \), \( \lambda \), \( \gamma \), \( \eta \), \( \zeta \) be the eigenvalues of \( X \), and \( u \), \( b \) the \( k \)-th coordinate of \( \nu \).

Then \( |u(\nu)| \leq \sqrt{n}. \)

Proof: Write \( \lambda = \zeta + \eta \zeta \). Then setting \( \eta = n^{-1/2} \), we have

\[
C \geq \text{Im} \sum_{x=1}^{n} z(x) \text{Re} \sum_{y=1}^{n} \text{Im} \sum_{z=1}^{n} (\lambda + \eta \zeta) \text{d}z \text{d}y \text{d}x
\]

\[
= \text{Im} \sum_{y=1}^{n} \sum_{z=1}^{n} \left( \sum_{x=1}^{n} x(y) \text{Im}(x(y)) \right. \text{d}x \text{d}y \text{d}z
\]

\[
= \sum_{y=1}^{n} \sum_{z=1}^{n} \text{Im}(y(y)) \theta_2
\]

\[
\geq \frac{1}{\eta^2} |u(\nu)|^2 \quad \text{(because of \( \eta \))}
\]

Then \( |u(\nu)| \leq \sqrt{n}. \)

Problem 4: (a) Let \( X \in \mathbb{R}^{n \times n} \) be a Wigner matrix s.t. \( \mathbb{E}[x_{ij}] = 0 \) and \( |x_{ij}| \leq \frac{1}{n} \).

Let \( Y \in \mathbb{R}^{n \times n} \) be a Wigner matrix with \( y_{ij} \sim \mathcal{N}(0, \frac{1}{n^{1/2}}). \)

(Note: \( Y \) is not normalized by \( n^{-1/2} \).

Show \( \mathbb{E}[\text{Tr}(X^* Y)] \leq \frac{1}{n} \mathbb{E}[\text{Tr}(X^* X)] \).

(b) Suppose \( X \in \mathbb{R}^{n \times n} \) is the centered and scaled adjacency matrix of an Erdős-Rényi graph with edge prob \( p \).

\[ X_{ij} = 0, \quad y_{ij} \sim \mathcal{N}(0, \frac{1}{n^{1/2} p}). \]

Apply part (a) to prove an upper bound for \( \|X\|_2 \).

For what \( p \to 0 \) can you show \( \|X\|_2 \to 2 \) as?