Spectral norm

Let $X \in \mathbb{R}^{n\times n}$ be Wigner, μ_n its ESD, μ the semicircle law on $[-2,2]$.

The statement $\mu_n \to \mu$ weakly means $\mu_n([a,b]) \to \mu([a,b])$ for any fixed interval $[a,b]$. I.e.,

$$|\{\lambda_i(X) : \lambda_i(X) \in [a,b]\}| = n \cdot \mu([a,b]) + o(n)$$

It does <u>not</u> address the following questions:

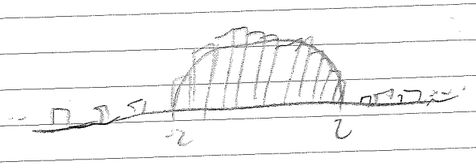
(1) For a shrinking interval $[a_n, b_n]$ with $|b_n - a_n| \to 0$, do we have

$$|\{\lambda_i(X) : \lambda_i(X) \in [a_n, b_n]\}| \approx n \cdot \mu([a_n, b_n])\ ?$$

(2) For an interval $[a,b]$ <u>outside</u> the support $[-2,2]$ of μ, are there in fact no eigenvalues of X in $[a,b]$?

(Weak convergence $\mu_n \to \mu$ only implies the number of eigs is $o(n)$.)

Check via simulation: If $X_{ij} = \frac{1}{\sqrt{n}} Z_{ij}$ where $\mathbb{E}[Z_{ij}] = 0$, $\mathbb{E}[Z_{ij}^2] = 1$, but $\mathbb{E}[|Z_{ij}|^3] = \infty$, then we get:



$\mu_n \to \mu$ weakly but $\|X\| \not\to 2$.

The Stieltjes transform method can address both (1) and (2). The moment method can address (2).

Moment method idea: For even integers k,

$$\|X\|^k = \max_i |\lambda_i|^k \le \sum_i \lambda_i^k = \operatorname{Tr} X^k \le n \cdot \|X\|^k.$$

So $\|X\| \le (\operatorname{Tr} X^k)^{1/k} \le n^{1/k} \cdot \|X\|$. If $k = C \log n$, then $n^{1/k} = e^{1/C}$, and $(\operatorname{Tr} X^k)^{1/k}$ is an upper bound for $\|X\|$ that is tight up to a small multiplicative factor for large C. [Furedi, Komlos '81]

Thm: Suppose X is a Wigner matrix, $X_{ij} = \frac{1}{\sqrt{n}} Z_{ij}$ where $\mathbb{E}[|Z_{ij}|^k] \le k^{\alpha k}$ for some constant $\alpha > 0$ and any $k > 0$. Then $\|X\| \to 2$ a.s.

Proof: Since $\mu_n \to \mu$ weakly a.s., this implies X has $c(\varepsilon)n$ eigenvalues in any interval $[2-\varepsilon, 2]$. So $\liminf_{n\to\infty} \|X\| \ge 2 - \varepsilon$ a.s., hence $\liminf_{n\to\infty} \|X\| \ge 2$ a.s.

For the upper bound: Fix $\varepsilon > 0$. Suppose we can show

(*) $\mathbb{E}[\operatorname{Tr} X^k] \le (2 + \frac{\varepsilon}{2})^k$ for some even $k \ge C \log n$ and all large n,

where $C \cdot \log \frac{2+\varepsilon}{2+\varepsilon/2} > 2$. Then

$$\begin{aligned}
\mathbb{P}[\|X\| \ge 2+\varepsilon] &= \mathbb{P}[\|X\|^k \ge (2+\varepsilon)^k] \\
&\le \frac{\mathbb{E}[\|X\|^k]}{(2+\varepsilon)^k} \\
&\le \frac{\mathbb{E}[\operatorname{Tr} X^k]}{(2+\varepsilon)^k} \le \left(\frac{2+\varepsilon/2}{2+\varepsilon}\right)^k = e^{-k \log \frac{2+\varepsilon}{2+\varepsilon/2}} < e^{-2\log n} = \frac{1}{n^2}.
\end{aligned}$$

Then by Borel-Cantelli, $\limsup_{n\to\infty} \|X\| \le 2 + \varepsilon$ a.s. Since $\varepsilon > 0$ is arbitrary, this yields $\limsup_{n\to\infty} \|X\| \le 2$ a.s.

It remains to show (*). Recall

$$\mathbb{E}[\operatorname{Tr} X^k] = \mathbb{E}\Big[\sum_{i_1,\dots,i_k=1}^{n} X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_k i_1}\Big]$$

$$= \sum_{\substack{\text{paths} \\ (i_1,\dots,i_k)}} \mathbb{E}[X_{i_1 i_2} \cdots X_{i_k i_1}],$$

where $(i_1, \dots, i_k) \longleftrightarrow$ path $i_1 \to i_2 \to i_3 \to \dots \to i_1$ on graph w/ n vertices.

For a path γ, let $v(\gamma) = \#$ distinct visited vertices

$e(\gamma) = \#$ distinct traversed (undirected) edges

Recall $v(\gamma) \le e(\gamma)+1$ since γ walks over a connected graph.

First, bound $\mathbb{E}[X_{i_1 i_2} \cdots X_{i_{k-1} i_1}]$:

- If some edge is traversed only once, then some X_{ij} appears only once in this product, so this is 0.
- More generally, let $k_1, \dots, k_{e(\gamma)}$ be the numbers of times the distinct edges are traversed, and suppose each $k_i \ge 2$. Then

$$\mathbb{E}[X_{i_1 i_2} \cdots X_{i_{k-1} i_1}] = \frac{1}{n^{k/2}} \mathbb{E}[Z_{i_1 i_2} \cdots Z_{i_{k-1} i_1}]$$

$$\le \frac{1}{n^{k/2}} \cdot k_1^{\alpha k_1} \cdots k_{e(\gamma)}^{\alpha k_{e(\gamma)}}$$

$$\le \frac{1}{n^{k/2}} \cdot k^{\alpha(k_1 + \dots + k_{e(\gamma)})} = \frac{1}{n^{k/2}} k^{\alpha k}.$$

Actually, we'll need a tighter bound: If edge i is not a self-loop and $k_i = 2$, then we may replace $k_i^{\alpha k_i}$ by 1. So in fact

$$\mathbb{E}[X_{i_1 i_2} \cdots X_{i_{k-1} i_1}] \le \frac{1}{n^{k/2}} \prod_{i=1}^{e(\gamma)} k_i^{\alpha k_i \cdot \mathbb{1}\{\text{edge } i \text{ is self-loop or } k_i \ge 3\}}$$

Let
$$p = \sum_{i=1}^{e(\gamma)} k_i \cdot \mathbb{1}\{\text{edge } i \text{ is self-loop}\}$$
$$q = \sum_{i=1}^{e(\gamma)} k_i \, \mathbb{1}\{\text{edge } i \text{ not self-loop}, k_i \ge 3\}$$
These are the total numbers of traversals of such edges.

Then $\mathbb{E}[X_{i_1 i_2} \cdots X_{i_{k-1} i_1}] \le \frac{1}{n^{k/2}} \cdot k^{\alpha(p+q)}$.

We'll want an upper bound on $p+q$ in terms of $\frac{k}{2}+1-v(\gamma)$.

Two observations:

(1) Let $\tilde{e}(\gamma)$ be number of distinct <u>non-self-loop</u> edges. Then
$$v(\gamma) \le \tilde{e}(\gamma)+1$$

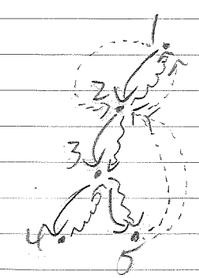
(2) By a pigeon-hole argument, if each edge is traversed at least twice,
$$\tilde{e}(\gamma) \le \frac{q}{3} + \frac{k-p-q}{2}$$
$$= \frac{k}{2} - \frac{p}{2} - \frac{q}{6} \le \frac{k}{2} - \frac{p+q}{6}.$$

Then $p+q \le 6\left(\frac{k}{2} - \tilde{e}(\gamma)\right) \le 6\left(\frac{k}{2} - v(\gamma) + 1\right)$.

We get $\mathbb{E}[X_{i_1 i_2} \cdots X_{i_{k-1} i_1}] \le \frac{1}{n^{k/2}} \cdot k^{6\alpha\left(\frac{k}{2} - v(\gamma)+1\right)}$.

Second, count paths γ that visit $v(\gamma)$ vertices, and traverse each edge at least twice:

- Corresponding to each path is a "canonical path" that is equivalent up to a permutation of the labels $\{1, \dots, n\}$, and where the distinct vertices visited are $1, 2, \dots, v(\gamma)$ in visited order.



Each canonical path is equivalent to exactly $n(n-1)(n-2)\cdots(n-v(\gamma)+1) \le n^{v(\gamma)}$ paths.

- For a canonical path, divide the steps into three cases:
  - $+$: This step visits a new vertex not previously visited.
  - $-$: This step is the second traversal (in either direction) of an edge corresponding to a previous $+$
  - $*$: Everything else (this case includes all the self-loops).

We may describe a canonical path by first describing the sequence of $+$, $-$, and $*$ steps:

$$+ + + - + * - * - - * *$$

Then providing vertex labels compatible with the above steps:

$$2\ 3\ 4\ 3\ 5\ 2\ 1\ 2\ 3\ 5\ 2\ 1.$$

Note: The sequence of $+$, $-$, and $*$ does not uniquely specify a sequence of compatible vertex labels. However, we have the following observations:

- Each $+$ does have a unique vertex label, b/c the path is canonical. (These must be $2, 3, 4, \ldots, v(\gamma)$ in order.)
- These steps $+$ form a tree on the vertices $1, 2, \ldots, v(\gamma)$.
- Before the first $*$, all $-$ labels are uniquely specified.
- If, in addition, we specify the label of each $*$ and the step preceding each $*$ (i.e. what are the two vertices that the $*$ step jumped between), then all remaining labels are uniquely specified.

  This is because each chain of consecutive $-$'s is a connected path of distinct edges on the tree. If we specify the start and end vertices of this path, then the path is unique.

So given a sequence of $+$, $-$, and $*$, the number of labelings is at most $v(\gamma)^{2(\#\text{ of } *\text{'s})} \le k^{2(\#\text{ of } *\text{'s})}$.

There are exactly $v(\gamma)-1$ $+$ steps, so also $v(\gamma)-1$ $-$ steps. Then $(\#\text{ of } *\text{'s}) = k - 2(v(\gamma)-1) = 2(\frac{k}{2} - v(\gamma) + 1)$. Then number of labelings is $\le k^{4(\frac{k}{2} - v(\gamma)+1)}$.

Number of sequences of $+$, $-$, and $*$ is

$$\le \binom{k}{\#\text{ of } *\text{'s}} \cdot 2^{k-(\#\text{ of } *\text{'s})} \le k^{2(\frac{k}{2} - v(\gamma)+1)} \cdot 2^k.$$

Total number of paths visiting $v(\gamma)$ vertices is $\le n^{v(\gamma)} \cdot 2^k \cdot k^{6(\frac{k}{2} - v(\gamma)+1)}$.

Combining these, and noting $1 \le v(\gamma) \le \frac{k}{2}+1$ for paths visiting each edge at least twice,

$$\begin{aligned}
\mathbb{E}[\operatorname{Tr} X^k] &= \sum_{v=1}^{\frac{k}{2}+1} \sum_{\gamma : v(\gamma)=v} \mathbb{E}[X_{i_1 i_2} \cdots X_{i_k i_1}] \\
&\le \sum_{v=1}^{\frac{k}{2}+1} n^v \cdot 2^k \cdot k^{6(\frac{k}{2}-v+1)} \cdot \frac{1}{n^{k/2}} \cdot k^{6\alpha(\frac{k}{2}-v+1)} \\
&= \sum_{v=1}^{\frac{k}{2}+1} n \cdot 2^k \cdot \left(\frac{k^{6+6\alpha}}{n}\right)^{\frac{k}{2}-v+1} \\
&\le n \cdot 2^k \sum_{j=0}^{\infty} \left(\frac{k^{8+6\alpha}}{n}\right)^j \le 2n \cdot 2^k,
\end{aligned}$$

for any $k \sim C \log n$ and $n \ge n_0(C)$ large enough.

Note $2n \cdot 2^k = e^{\log n + (k+1)\log 2} < e^{k \cdot \log(2+\varepsilon)} = (2+\varepsilon)^k$

for $k \sim C \log n$ and C a large enough constant. This shows $(*)$, as desired.

Overview of local semicircle law

Notation: We write $X(z) \prec \Psi(z)$ if, for any $\varepsilon, D > 0$, there is $n_0(\varepsilon, D) > 0$ s.t. $\mathbb{P}[|X(z)| > n^{\varepsilon}\Psi(z)] \le n^{-D}$ for all $n \ge n_0(\varepsilon, D)$.

"For all large n, $|X(z)|$ is not much larger than $\Psi(z)$ with high probability."

We also write this as $X(z) = O_{\prec}(\Psi(z))$.

We'll say this holds uniformly over $z \in D$ if

$$\mathbb{P}[\exists z \in D : |X(z)| > n^{\varepsilon}\Psi(z)] \le n^{-D} \text{ for all } n \ge n_0(\varepsilon, D).$$

Thm (Local semicircle law): Let $X \in \mathbb{R}^{n \times n}$ be a Wigner matrix s.t. $\mathbb{E}[|\sqrt{n}X_{ij}|^k] \le C_k$ for some constants $C_k > 0$ and all $k \ge 0$.

Fix any constants $C, \tau > 0$, and set

$$D = \{z \in \mathbb{C}^+ : \operatorname{Re} z \in [-C, C],\ \operatorname{Im} z \in [n^{-1+\tau}, C]\}.$$

Let $G_n(z) = (X - zI)^{-1}$, $m_n(z) = \frac{1}{n}\operatorname{Tr} G_n(z)$, and $m(z)$ the Stieltjes transform of the semicircle law. Write $\eta = \operatorname{Im} z$ and $\kappa = \min(|\operatorname{Re} z - 2|, |\operatorname{Re} z + 2|)$. Then:

(a) Uniformly over $z \in D$, $|m_n(z) - m(z)| \prec \frac{1}{n\eta}$

(b) Uniformly over $\{z \in D : \operatorname{Re} z \notin [-2, 2]\}$

$$|m_n(z) - m(z)| \prec \frac{1}{n(\eta + \kappa)} + \frac{1}{(n\eta)^2\sqrt{\eta + \kappa}}$$

(c) Uniformly over $z \in D$, for all $i \in \{1, \dots, n\}$,

$$|G_n(z)_{ii} - m(z)| \prec \sqrt{\frac{\operatorname{Im} m(z)}{n\eta}} + \frac{1}{n\eta}$$

and for all $i \ne j \in \{1, \dots, n\}$,

$$|G_n(z)_{ij}| \prec \sqrt{\frac{\operatorname{Im} m(z)}{n\eta}} + \frac{1}{n\eta}.$$

Proof: This is a (non-trivial) generalization of the Schur complement argument from Lecture 3. See Erdos, Knowles, Yau, Yin, 2013 "The Local Semicircle Law for a General Class of Random Matrices." A full exposition is also given in Benaych-Georges, Knowles 2018 "Lectures on the local semicircle law for Wigner matrices."

Remarks: (a) and (b) give bounds on $|m_n(z) - m(z)|$ for $\operatorname{Im} z$ approaching the real line at a rate $n^{-1+\tau}$. We'll see that implies approximation of μ_n by μ over intervals $[a_n, b_n]$ with $|b_n - a_n| \gtrsim n^{-1+\tau}$.

(c) gives an entrywise bound on the resolvent, and shows $G(z) \approx m(z) \cdot I$ in an entrywise sense. We'll see that this is useful for studying <u>eigenvectors</u> of X.

Note that the density of the semicircle law is bounded — hence so is $\operatorname{Im} m(z)$ over $z \in D$. So (c) implies the simpler bound.

$$|G_n(z)_{ii} - m(z)|,\ |G_n(z)_{ij}| \prec \frac{1}{\sqrt{n\eta}}$$

We get more refined bounds by using (see Problem 2)

$$\operatorname{Im} m(z) \asymp \begin{cases} \sqrt{\kappa + \eta} & \operatorname{Re} z \in [-2, 2] \\ \frac{\eta}{\sqrt{\kappa + \eta}} & \operatorname{Re} z \notin [-2, 2] \end{cases}$$

Proposition: Suppose, at some $\lambda + i\eta$, that $2n\eta \cdot \operatorname{Im} m_n(\lambda + i\eta) < 1$. Then μ_n has no eigenvalues in the interval $[\lambda - \eta, \lambda + \eta]$.

Proof: We have

$$\operatorname{Im} m_n(\lambda+i\eta) = \operatorname{Im} \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\lambda_j(X)-\lambda-i\eta}$$

$$= \frac{1}{n}\sum_{j=1}^{n} \frac{\eta}{(\lambda_j(X)-\lambda)^2+\eta^2}$$

$$\geq \frac{1}{n}\sum_{j:|\lambda_j(X)-\lambda|\leq\eta} \frac{\eta}{(\lambda_j(X)-\lambda)^2+\eta^2}$$

$$\geq \frac{1}{n}\cdot\left|\{j:|\lambda_j(X)-\lambda|\leq\eta\}\right|\cdot\frac{1}{2\eta}$$

Thus $2n\eta\cdot\operatorname{Im} m_n(\lambda+i\eta) \geq$ #eigenvalues in $[\lambda-\eta,\lambda+\eta]$.

If this is <1, then the eigenvalues must be 0.

Cor: For any constants $C,\varepsilon>0$, X has no eigenvalues in $[2+\varepsilon, C]$ a.s. for all large n.

Proof: Set $\eta = n^{-2/3}$. Consider

$$L = \{z\in\mathbb{C}^+ : \operatorname{Im} z=\eta,\ \operatorname{Re} z\in[2+\varepsilon, C]\}.$$

Uniformly over $z\in L$:

$$\operatorname{Im} m(z) \lesssim \eta \lesssim n^{-2/3}$$

$$|m_n(z)-m(z)| < \frac{1}{n}+\frac{1}{(n\eta)^2} \lesssim n^{-2/3}.$$

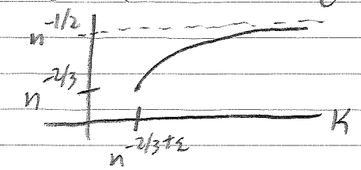
Then $\mathbb{P}[\exists z\in L: \operatorname{Im} m_n(z) > n^{-2/3+\delta}] < n^{-D}$ for any $\delta, D>0$ and all large n. On this event, $2n\eta\cdot\operatorname{Im} m_n(z) \leq 2n^{-1/3+\delta} < 1$ for all $z\in L$, so there are no eigenvalues in $[2+\varepsilon-\eta, C+\eta]$.

Combining with a bound $\|X\|\leq C$ w.h.p., we get $\|X\|\to 2$ a.s.

Cor: For any constants $C,\varepsilon>0$, X has no eigenvalues in $[2+n^{-2/3+\varepsilon}, C]$ a.s. for all large n.

Proof: Consider

$$L = \{z = 2+K+i\eta : K\in[n^{-2/3+\varepsilon}, C-2],\ \eta = n^{-1/2-\varepsilon/4}K^{1/4}\}.$$



So $K = \eta^4\cdot n^{2+\varepsilon}$.

Uniformly over $z\in L$:

$$\operatorname{Im} m(z) \lesssim \frac{\eta}{\sqrt{K+\eta}} \leq \frac{\eta}{\sqrt{K}} = \frac{\eta}{\eta^2\cdot n^{1+\varepsilon/2}} = \frac{1}{n\eta}\cdot n^{-\varepsilon/2}.$$

$$|m_n(z)-m(z)| < \frac{1}{n(\eta+K)} + \frac{1}{(n\eta)^2\sqrt{K+\eta}}$$

$$\lesssim \frac{1}{nK} + \frac{1}{n^2\eta^2\sqrt{K}} \lesssim \frac{1}{n^{3+\varepsilon/2}\eta^4} \lesssim \frac{1}{n\eta}\cdot n^{-\varepsilon/2}$$

Uses $\eta^2 \geq n^{-2/3}$.

So also $2n\eta\cdot\operatorname{Im} m_n(z) < 1$ w.h.p. for all $z\in L$.

Proposition: (a) (Cauchy integral) Suppose μ has bounded support and Stieltjes transform $m(z)$. Let $f:\mathbb{R}\to\mathbb{R}$, and suppose f extends analytically to a domain $U\subseteq\mathbb{C}$ containing $\operatorname{supp}\mu$. Then for any (counterclockwise) contour $\Gamma\subset U$ enclosing $\operatorname{supp}\mu$,

$$\int f(\lambda)\,d\mu(\lambda) = -\frac{1}{2\pi i}\oint_\Gamma f(z)\,m(z)\,dz.$$

(b) (Helffer–Sjöstrand) Suppose $f:\mathbb{R}\to\mathbb{R}$ is $(k+1)$-times continuously differentiable, w/ bounded support. Define an "almost analytic" extension by

$$g(x+iy) = \chi(y)\cdot\left(f(x)+iyf'(x)+\frac{(iy)^2}{2}f''(x)+\dots+\frac{(iy)^k}{k!}f^{(k)}(x)\right),$$

where $\chi(y)$ is smooth, $\equiv 1$ in a neighborhood of 0, symmetric around 0, compact support.

Then

$$\int f(\lambda)\,d\mu(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[(\partial_x+i\partial_y)g(x+iy)\right]\cdot m(x+iy)\,dx\,dy.$$

Proof: For (a), Cauchy integral formula gives

$$f(\lambda) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z-\lambda}\, dz$$

for any λ in the contour. Then $\int f(\lambda)\, d\mu(\lambda) = \frac{1}{2\pi i} \oint \int \frac{f(z)}{z-\lambda}\, d\mu(\lambda)\, dz$

$$= -\frac{1}{2\pi i} \oint f(z) m(z)\, dz.$$

For (b), an application of Green's Theorem gives

$$f(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\partial_x + i\partial_y)(g(x+iy))}{\lambda - (x+iy)}\, dx\, dy.$$

See [AGZ, Proof of Lemma 5.5.5]. Then (b) follows similarly.

Cor: Uniformly over (n-dependent) intervals $I \subset [-C, C]$,

$$\left| \frac{1}{n} \left| \{ \lambda_i(X) : \lambda_i(X) \in I \} \right| - \mu(I) \right| < n^{-1}$$

where μ is the semicircle law.

Proof: Let μ_n be the ESD of X. Fix $\eta = n^{-1+\tau}$. Define $f : \mathbb{R} \to [0,1]$ s.t.

- $f(x) = 1$ for $x \in I$
- $f(x) = 0$ for $\operatorname{dist}(x, I) \geq \eta$
- $|f'(x)| \leq \frac{C}{\eta}$, $|f''(x)| \leq \frac{C}{\eta^2}$.

Define $\chi : \mathbb{R} \to [0,1]$, $\chi(y) = 1$ for $y \in [-1,1]$, $\chi(y) = 0$ for $y \notin [2,2]$, $|\chi'(y)| \leq C$ for all y. Apply Helffer-Sjöstrand with $k=1$:

$$\int f(x)\, d\mu_n(x) - \int f(x)\, d\mu(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[(\partial_x + i\partial_y)\left(\chi(y)(f(x) + iy f'(x)) \right) \right] \cdot \left(m_n(x+iy) - m(x+iy) \right) dx\, dy.$$

Note: $\partial_x f(x) = -i\partial_y (y f'(x))$. Then

$$(\partial_x + i\partial_y)\left(\chi(y)(f(x) + iy f'(x)) \right) = iy\, \chi(y) \cdot f''(x) + i\chi'(y)(f(x) + iy f'(x)).$$

We have $\chi'(y) = 0$ unless $|y| \in [1,2]$. For such y, $|m_n(x+iy) - m(x+iy)| < \frac{1}{n}$. Then

$$\left| \int_{\mathbb{R}^2} \chi'(y)(f(x) + iy f'(x))(m_n(x+iy) - m(x+iy))\, dx\, dy \right|$$

$$< \frac{1}{n} \int_{\mathbb{R}} (|f(x)| + |f'(x)|)\, dx < \frac{1}{n}.$$

For the first term, consider first

$$I := \int_\eta^\infty \int_{\mathbb{R}} iy\, \chi(y) f''(x) \cdot (m_n(x+iy) - m(x+iy))\, dx\, dy.$$

Integrate by parts first over x, then over y:

$$I = -\int_\eta^\infty \int_{\mathbb{R}} iy\, \chi(y) \cdot f'(x)\, \partial_x (m_n(x+iy) - m(x+iy))\, dx\, dy$$

$$= -\int_\eta^\infty \int_{\mathbb{R}} y\, \chi(y) f'(x) \cdot \partial_y (m_n(x+iy) - m(x+iy))\, dx\, dy$$

$$= +\int_\eta^\infty \int_{\mathbb{R}} (\chi(y) + y\chi'(y)) \cdot f'(x) \cdot (m_n(x+iy) - m(x+iy))\, dx\, dy$$

$$- \eta\, \chi(\eta) \int_{\mathbb{R}} f'(x)(m_n(x+i\eta) - m(x+i\eta))\, dx.$$

- $\left| \int_\eta^\infty \int_{\mathbb{R}} y\, \chi'(y) f'(x)(m_n(x+iy) - m(x+iy))\, dx\, dy \right| < \frac{1}{n}$ by same argument as above
- $\left| \eta\, \chi(\eta) \int_{\mathbb{R}} f'(x)(m_n(x+i\eta) - m(x+i\eta))\, dx \right| < \frac{\eta\, \chi(\eta) \int |f'(x)|\, dx}{n\eta} < \frac{1}{n}.$
- $\left| \int_\eta^\infty \int_{\mathbb{R}} \chi(y) f'(x)(m_n(x+iy) - m(x+iy))\, dx\, dy \right|$

$$< \int_{\mathbb{R}} f'(x) \cdot \left(\int_\eta^2 \frac{1}{ny}\, dy \right) dx < \frac{\log \eta}{n} < \frac{1}{n}.$$

So $|I| < \frac{1}{n}$.

Consider next

$$II := \int_0^\eta \int_{\mathbb{R}} iy\, \chi(y) f''(x)(m_n(x+iy) - m(x+iy))\, dx\, dy$$

Then $\mathrm{Re}\, \mathrm{II} = \int_0^\eta \int_{\mathbb{R}} y\chi(y) f''(x) (\mathrm{Im}\, m(x+iy) - \mathrm{Im}\, m_n(x+iy))\, dx\, dy$.

Apply $0 \le y \cdot \mathrm{Im}\, m(x+iy) = \int \frac{y^2}{(x-\lambda)^2+y^2} d\mu(\lambda) \le \int \frac{\eta^2}{(x-\lambda)^2+\eta^2} d\mu(\lambda) = \eta \cdot \mathrm{Im}\, m(x+i\eta)$,

and similarly for $y \cdot \mathrm{Im}\, m_n(x+i\eta)$. Then

$$|\mathrm{Re}\, \mathrm{II}| \le \int_0^\eta \int_{\mathbb{R}} \chi(y) |f''(x)| \cdot (\eta\, \mathrm{Im}\, m(x+i\eta) + \eta\, \mathrm{Im}\, m_n(x+i\eta))\, dx\, dy$$

$$\lesssim \eta^2 \int_{\mathbb{R}} |f''(x)| dx \lesssim \eta = n^{-1+\tau}.$$

We have $\left|\int f\, d\mu_n - \int f\, d\mu\right| = \left|\mathrm{Re}\left(\int f\, d\mu_n - \int f\, d\mu\right)\right| \lesssim n^{-1+\tau}$.

Here $\tau > 0$ is arbitrary, so $\left|\int f\, d\mu_n - \int f\, d\mu\right| \lesssim \frac{1}{n}$.

Finally, denoting $f = f_I$, $I' = \{x \in I : \mathrm{dist}(x, \partial I) \ge \eta\}$,

$$\int f_{I'}\, d\mu_n(x) \le \frac{1}{n} |\{\lambda_i(X) : \lambda_i(X) \in I\}| \le \int f_I\, d\mu_n(x),$$

Applying the above to I and I', and $\int_{I \setminus I'} d\mu(x) \lesssim \eta = n^{-1+\tau}$ yields the result.

Cor: Let $u_1, \dots, u_n$ be the eigenvectors of X, and $u_i(k)$ the kth coordinate of u_i.

Then $|u_i(k)|^2 \lesssim n^{-1}$.

Proof: W.h.p., $\lambda_i \in [-C, C]$. Then, setting $\eta = n^{-1+\tau}$, w.h.p.

$$C \ge \sup_{z \in D} \mathrm{Im}\, G_n(z)_{kk} \ge \mathrm{Im}\, G_n(\lambda_i + i\eta)_{kk}$$

$$= \mathrm{Im}\, e_k^T \left(\sum_{j=1}^n \frac{1}{\lambda_j - \lambda_i - i\eta} u_j u_j^T\right) e_k$$

$$= \sum_{j=1}^n \frac{\eta}{(\lambda_j - \lambda_i)^2 + \eta^2} |u_j(k)|^2$$

$$\ge \frac{1}{\eta} |u_i(k)|^2 \quad \text{(restrict to } j = i\text{)}.$$

Then $|u_i(k)|^2 \le C\eta = Cn^{-1+\tau}$.

<u>Problem 4</u> (a) Let $X \in \mathbb{R}^{n \times n}$ be a Wigner matrix s.t. $\mathbb{E}[X_{ij}^2] \le \frac{1}{n}$ and $|X_{ij}| \le \frac{1}{q_n}$ for all i, j. Fix any even integer k, and let $r = \lceil q_n^2 \rceil + k$.

Let $Y \in \mathbb{R}^{r \times r}$ be a Wigner matrix with $Y_{ij} \sim \frac{\text{Bernoulli}(\frac{1}{2}) - \frac{1}{2}}{\sqrt{\frac{1}{4}(1-\frac{1}{4})}}$.

(Note: Y is not normalized by $\frac{1}{\sqrt{r}}$.)

Show $\quad q_n^k\, \mathbb{E}[\mathrm{Tr}\, X^k] \le \frac{n}{r} \cdot \mathbb{E}[\mathrm{Tr}\, Y^k]$.

(b) Suppose $X \in \mathbb{R}^{n \times n}$ is the centered and scaled adjacency matrix of an Erdos-Renyi graph with edge prob p_n:

$$X_{ii} = 0, \qquad X_{ij} \sim \frac{\text{Bernoulli}(p_n) - p_n}{\sqrt{n} \cdot \sqrt{p_n(1-p_n)}} \quad \text{for } i \ne j.$$

(So $\mathbb{E}[X_{ij}^2] = \frac{1}{n}$, $|X_{ij}| \le \frac{1}{\sqrt{np_n}}$.)

Apply part (a) to prove an upper bound for $\|X\|$.

For what $p_n \to 0$ can you show $\|X\| \to 2$ a.s.?