# Free Probability

Let $X \in \mathbb{R}^{n \times n}$ be Wigner, $A \in \mathbb{R}^{n \times n}$ symmetric, deterministic.

If $A$ has fixed rank $k$, eigenvalues of $X+A$ are semicircle plus outliers.

If $A$ has full rank, the bulk eigenvalue distribution of $X+A$ changes. We saw this in Lecture 4. What about other matrices, e.g. $AXA$? $AX^2A+X$?

Example: Write $n = N+M$. Set

$$A_1 = \begin{pmatrix} Id_N & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & Id_M \end{pmatrix}.$$

Decompose the Wigner matrix $X$ as

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \quad \text{where } X_{12} = X_{21}^T. \quad (X_{12} \in \mathbb{R}^{N \times M} \text{ has independent entries.})$$

Then

$$A_1 X A_2 \cdot A_2 X A_1 = \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X_{21} & 0 \end{pmatrix} = \begin{pmatrix} X_{12}X_{21} & 0 \\ 0 & 0 \end{pmatrix}$$

has the same eigenvalues, plus $M$ $0$'s, as the sample covariance matrix $X_{12} \cdot X_{21}$.

By understanding general polynomials of simpler matrices, we can understand more complex matrix models.

Intuition: The eigenvectors of $X$ and $A$ are with high probability "unaligned" or in "generic position." This is easiest to see when $X \sim GOE(n)$, but also true for general Wigner $X$.

One way to describe this: Let $\operatorname{tr} M = \mathbb{E}[\frac{1}{n} \operatorname{Tr} M]$.

Suppose $P_1, \dots, P_k$ are polynomials s.t. $\operatorname{tr} P_j(X) \to 0$ for each $j$, as $n \to \infty$,

$Q_1, \dots, Q_k$ are polynomials s.t. $\operatorname{tr} Q_j(A) \to 0$ for each $j$, as $n \to \infty$.

Then $\operatorname{tr} P_1(X) Q_1(A) P_2(X) Q_2(A) \dots P_k(X) Q_k(A) \to 0$ as $n \to \infty$.

If $X, A$ satisfy this for all such $P_1, \dots, P_k, Q_1, \dots, Q_k$ and fixed $k \geq 1$, they are called asymptotically free.

Calculation of moments: Write as shorthand $\overline{M} = \operatorname{tr} M \cdot I$.

$$\operatorname{tr} X+A = \operatorname{tr} X + \operatorname{tr} A.$$

$$\operatorname{tr} (X+A)^2 = \operatorname{tr} X^2 + \operatorname{tr} XA + \operatorname{tr} AX + \operatorname{tr} A^2$$

$$= \operatorname{tr} X^2 + 2\underbrace{\operatorname{tr} XA} + \operatorname{tr} A^2$$

$$= \operatorname{tr} X\overline{A} + \underbrace{\operatorname{tr} X(A-\overline{A})}_{\to 0, \text{ by freeness since } \operatorname{tr} X = 0,\ \operatorname{tr} A - \overline{A} = 0}$$

$$= \operatorname{tr} X \cdot \operatorname{tr} A + o(1)$$

$$= \operatorname{tr} X^2 + 2 \operatorname{tr} X \cdot \operatorname{tr} A + \operatorname{tr} A^2 + o(1)$$

$$\operatorname{tr} (X+A)^3 = \operatorname{tr} X^3 + 3\underbrace{\operatorname{tr} X^2 A} + 3 \operatorname{tr} XA^2 + \operatorname{tr} A^3$$

$$= \operatorname{tr} \left[ (X^2 - \overline{X^2})(A - \overline{A}) + \overline{X^2}(A - \overline{A}) + (X^2 - \overline{X^2})\overline{A} + \overline{X^2}\,\overline{A} \right]$$

$$= \operatorname{tr} X^2 \cdot \operatorname{tr} A + o(1)$$

Similarly $\operatorname{tr} XA^2 = \operatorname{tr} X \cdot \operatorname{tr} A^2 + o(1)$

$$= \operatorname{tr} X^3 + 3 \operatorname{tr} X^2 \operatorname{tr} A + \operatorname{tr} A^3 + o(1)$$

$$\operatorname{tr} (X+A)^4 = \operatorname{tr} X^4 + 4 \operatorname{tr} X^3 A + 2 \operatorname{tr} XAXA + 4 \operatorname{tr} X^2 A^2 + 4 \operatorname{tr} XA^3 + \operatorname{tr} A^4.$$

Apply $\operatorname{tr} XAXA = \operatorname{tr} \left[ X(A-\overline{A})X(A-\overline{A}) + X\overline{A}X(A-\overline{A}) + X(A-\overline{A})X\overline{A} + X\overline{A}X\overline{A} \right]$

$$= 2\,\mathrm{tr}A \cdot \mathrm{tr}\, X^2(A-\bar{A}) + (\mathrm{tr}A)^2 \cdot \mathrm{tr}X^2 + o(1)$$
$$= (\mathrm{tr}A)^2 \cdot \mathrm{tr}X^2 + o(1)$$

So $\mathrm{tr}(X+A)^4 = \mathrm{tr}X^4 + 2(\mathrm{tr}A)^2\,\mathrm{tr}X^2 + 4\,\mathrm{tr}X^2 \cdot \mathrm{tr}A^2 + \mathrm{tr}A^4 + o(1)$

This extends to an algorithm to compute $\lim_{n\to\infty} \mathrm{tr}\, P(X,A)$ for any polynomial in $X$ and $A$, in terms of $\lim_{n\to\infty} \mathrm{tr}\, X^j$ and $\lim_{n\to\infty} \mathrm{tr}\, A^k$.

Def: Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$ with multiplicative unit $1$. (So $a\cdot 1 = 1\cdot a = a$, $a(b+c) = a\cdot b + a\cdot c$, $(a\cdot b)^* = b^* a^*$, $a^{**} = a$, etc.) A function $\tau : \mathcal{A} \to \mathbb{C}$ is a <u>trace</u> if

- $\tau(a^*) = \overline{\tau(a)}$
- $\tau(c_1 a_1 + c_2 a_2) = c_1 \tau(a_1) + c_2 \tau(a_2)$ for $c_1, c_2 \in \mathbb{C}$, $a_1, a_2 \in \mathcal{A}$.
- $\tau(1) = 1$
- $\tau(ab) = \tau(ba)$ (so $\tau(abc) = \tau(bca) = \tau(cab)$)

$(\mathcal{A}, \tau)$ is a <u>noncommutative probability space</u>.

Example: $\mathcal{A} =$ (random) matrices in $\mathbb{C}^{n\times n}$, $\tau = \mathrm{tr} = \mathbb{E} \circ \frac{1}{n}\mathrm{Tr}$.

Def: $\mathcal{A}_i \subseteq \mathcal{A}$ is a <u>subalgebra</u> if $a, b \in \mathcal{A}_i \Rightarrow a+b \in \mathcal{A}_i$, $a\cdot b \in \mathcal{A}_i$, $c\cdot a \in \mathcal{A}_i$ $\forall c \in \mathbb{C}$, $a^* \in \mathcal{A}_i$. The subalgebra <u>generated by</u> $\{a_1, \dots, a_k\}$ is all polynomials in $\{a_1, \dots, a_k, a_1^*, \dots, a_k^*\}$.

Def: Subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_k \subseteq \mathcal{A}$ are <u>free</u> if $\tau(a_1 a_2 \dots a_m) = 0$ whenever:

- Each $a_i$ belongs to some $\mathcal{A}_{j(i)}$ and $\tau(a_i) = 0$, and
- No two consecutive $a_i$'s belong to the same $\mathcal{A}_{j(i)}$. I.e., $j(1) \neq j(2)$, $j(2) \neq j(3)$, $\dots$, $j(m-1) \neq j(m)$.

Collections of elements $A_1, \dots, A_m \subseteq \mathcal{A}$ are free if their generated subalgebras are. In particular, $a_1, a_2 \in \mathcal{A}$ are free if $\mathcal{A}_1 = \{\mathrm{poly}(a_1, a_1^*)\}$ and $\mathcal{A}_2 = \{\mathrm{poly}(a_2, a_2^*)\}$ are free.

This is somewhat analogous to <u>independence</u> of collections of random variables.

Def: Collections of (random) matrices $\mathcal{A}_1^{(n)}, \dots, \mathcal{A}_m^{(n)} \subseteq \mathbb{R}^{n\times n}$ are <u>asymptotically free</u> with respect to $\mathrm{tr} = \mathbb{E} \circ \frac{1}{n}\mathrm{Tr}$ if there exists a n.c.p. space $(\mathcal{A}, \tau)$ and corresponding $\mathcal{A}_1, \dots, \mathcal{A}_m \subseteq \mathcal{A}$ s.t. for every fixed polynomial $P$,
$$\mathrm{tr}\, P(\{\mathcal{A}_1^{(n)}, \dots, \mathcal{A}_m^{(n)}, \mathcal{A}_1^{(n)*}, \dots, \mathcal{A}_m^{(n)*}\}) \to \tau(P(\mathcal{A}_1, \dots, \mathcal{A}_m, \mathcal{A}_1^*, \dots, \mathcal{A}_m^*))$$
as $n \to \infty$, and $\mathcal{A}_1, \dots, \mathcal{A}_m$ are free in $(\mathcal{A}, \tau)$.

Thm: Let $W_1, \dots, W_m \in \mathbb{R}^{n\times n}$ be independent Wigner matrices, and $D_1, \dots, D_\ell \in \mathbb{C}^{n\times n}$ deterministic matrices such that $\frac{1}{n}\mathrm{Tr}\, P(D_1, \dots, D_\ell, D_1^*, \dots, D_\ell^*)$ has a limit as $n\to\infty$, for any fixed polynomial $P$. ($D_1, \dots, D_\ell$ have a limiting joint law) Then $W_1, \dots, W_m$, and $\mathcal{D}^{(n)} = \{D_1, \dots, D_\ell\}$ are asymptotically free. I.e., there exist $w_1, \dots, w_m, d_1, \dots, d_\ell \in \mathcal{A}$ where $w_1, \dots, w_m, \{d_1, \dots, d_\ell\}$ are free, and
$$\mathrm{tr}\, P(W_1, \dots, W_m, D_1, \dots, D_\ell, D_1^*, \dots, D_\ell^*) \to \tau(P(w_1, \dots, w_m, d_1, \dots, d_\ell, d_1^*, \dots, d_\ell^*))$$
for every fixed polynomial $P$.

Note: In particular, $\mathrm{tr}\, W_i^k \to \tau(w_i^k)$, so $\tau(w_i^k)$ must be the $k^{th}$ moment of the semicircle law for every $k$. This uniquely defines $\tau(P(w_i))$ for any univariate polynomial $P$. The elements $w_1, \dots, w_m$ are called <u>free semicircular elements</u> in $\mathcal{A}$.

Def: The <u>law</u> of $a \in \mathcal{A}$ is the collection of moments $\{\tau(w) : w \text{ any monomial in } a \text{ and } a^*\}$. More generally, the <u>(joint) law</u> of $\mathcal{A}_i \subseteq \mathcal{A}$ is the collection of joint moments $\{\tau(w) : w \text{ any monomial in elements of } \mathcal{A}_i\}$.

(The law of $a$ is *semicircular* if $a=a^*$ and $\tau(a^k) = k^{th}$ moment of semicircle law on $[-2,2]$ for each $k \geq 0$.)

Prop: Suppose $\mathcal{A}_1, \dots, \mathcal{A}_\ell \subseteq \mathcal{A}$ are free. Then the law of $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_\ell$ is uniquely determined by the individual laws of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_\ell$.

Proof: It suffices to show, for any $a_1, \dots, a_k$ where each $a_i$ belongs to (the subalgebra) $\mathcal{A}_{j(i)}$ and $j(1) \neq j(2), j(2) \neq j(3), \dots, j(k-1) \neq j(k)$, that $\tau(a_1 \dots a_k)$ is determined.

We induct on $k$. For $k=1$, $\tau(a_1)$ is clearly determined by the law of $\mathcal{A}_{j(1)}$.

Suppose this holds up to $k-1$. Write

$$\bar{a}_i = \tau(a_i) \cdot 1, \qquad a_i = \bar{a}_i + (a_i - \bar{a}_i).$$

Then $$\tau(a_1 \dots a_k) = \tau\left(\prod_{i=1}^{k} (\bar{a}_i + (a_i - \bar{a}_i))\right)$$
$$= \tau((a_1 - \bar{a}_1) \dots (a_k - \bar{a}_k)) + \tau(\text{other terms}).$$

Each other term has at least one $\bar{a}_i$, which can be pulled out of $\tau$ as a scalar factor $\tau(a_i)$. The remaining term can be written as $\tau(\tilde{a}_1 \dots \tilde{a}_{k'})$ for some $k' \leq k-1$, each $\tilde{a}_i$ belonging to a single one of $\mathcal{A}_1, \dots, \mathcal{A}_\ell$, so it is determined by the induction hypothesis.

The first term $\tau((a_1 - \bar{a}_1) \dots (a_k - \bar{a}_k)) = 0$ by the definition of freeness.

So $\tau(a_1 \dots a_k)$ is also determined, completing the induction.

Unraveling this induction gives an algorithm for computing $\tau(P(\{\mathcal{A}_1, \dots, \mathcal{A}_m\}))$ for any polynomial $P$ in terms of the individual laws of $\mathcal{A}_1, \dots, \mathcal{A}_m$.

Applying to independent Wigner $W_1, \dots, W_m$ and deterministic $D_1, \dots, D_\ell$, for any polynomial $P$, we have

$$\mathbb{E}\left[\frac{1}{n} \operatorname{Tr} P(W_1, \dots, W_m, D_1, \dots, D_\ell)\right] \to \tau(w_1, \dots, w_m, d_1, \dots, d_\ell),$$

and we may use this algorithm to compute $\tau(w_1, \dots, w_m, d_1, \dots, d_\ell)$ in terms of moments of the semicircle law and the limiting moments

$$\tau(Q(d_1, \dots, d_\ell)) = \lim_{n \to \infty} \frac{1}{n} \operatorname{Tr} Q(D_1, \dots, D_\ell)$$

of polynomials of the deterministic matrices $D_1, \dots, D_\ell$.

## Free cumulants, R-transform, and additive free convolution

Q: Suppose $a, b \in \mathcal{A}$ are free. How to compute $\tau((a+b)^k)$ in a simple way?

Analogy to classical probability: For a r.v. $X$, its moments are $m_k(X) = \mathbb{E}[X^k]$.

We define a moment-generating-function by

$$\varphi_X(t) = \mathbb{E}[e^{tX}] = 1 + t\mathbb{E}X + \frac{t^2}{2}\mathbb{E}X^2 + \frac{t^3}{6}\mathbb{E}X^3 + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!} m_k(X).$$

Related to the moments are a set of cumulants $\kappa_k$:

$\kappa_1(X) = m_1$ (mean)

$\kappa_2(X) = m_2 - m_1^2$ (variance)

$\kappa_3(X) = m_3 - 3m_2 m_1 + 2m_1^3$

$\kappa_4(X) = m_4 - 4m_3 m_1 - 3m_2^2 + 12 m_2 m_1^2 - 6m_1^4$.

These are the coefficients of $t$ in the cumulant generating function

$$\log \varphi_X(t) = \log \mathbb{E}[e^{tX}] \equiv \sum_{k=0}^{\infty} \frac{t^k}{k!} \kappa_k(X).$$

Fact: $$m_k(X) = \sum_{\substack{\text{partitions } \pi \\ \text{of } \{1,\dots,k\}}} \prod_{S \in \pi} \kappa_{|S|}(X)$$ (moment-cumulant relations)

Then $\kappa_k(X) = m_k(X) - \sum_{\pi:|\pi|\ge 2} \prod_{S\in\pi} \kappa_{|S|}(X)$. (*)

The cumulants on the right are $\kappa_1(X),\dots,\kappa_{k-1}(X)$, of lower order. So (*) may be used as an alternative recursive definition of $\kappa_k(X)$, in terms of moments $m_1(X),\dots,m_k(X)$.

More generally, define the mixed moment $m_k(X_1,\dots,X_k)=\mathbb{E}[X_1\cdots X_k]$, and the mixed cumulant via

$$m_k(X_1,\dots,X_k)=\sum_{\substack{\text{partitions }\pi\\ \text{of }\{1,\dots,k\}}} \prod_{S\in\pi}\kappa_{|S|}(X_S),$$

where $X_S=(X_i : i\in S)$. I.e.,

$$\kappa_k(X_1,\dots,X_k) = m_k(X_1,\dots,X_k) - \sum_{\pi:|\pi|\ge2}\prod_{S\in\pi}\kappa_{|S|}(X_S). \quad (**)$$

In particular, $\kappa_2(X_1,X_2)=m_2(X_1,X_2)-m_1(X_1)m_1(X_2)=\mathrm{Cov}[X_1,X_2]$.

We have $m_k(X)=m_k(X,\dots,X)$, $\kappa_k(X)=\kappa_k(X,\dots,X)$.

Fact: If $X, Y$ are independent, then all mixed cumulants of $X$ and $Y$ other than $\kappa_k(X,\dots,X)$ and $\kappa_k(Y,\dots,Y)$ for $k\ge1$ vanish.

E.g. $\kappa_2(X,Y)=0$, $\kappa_3(X,X,Y)=\kappa_3(X,Y,X)=0$, etc.

Note that $m_k$ is linear in each argument:

$$\mathbb{E}[(c_1X_1+c_2X_1')X_2\cdots X_k] = c_1\mathbb{E}[X_1\cdots X_k]+c_2\mathbb{E}[X_1'X_2\cdots X_k].$$

Then by (**) and induction on $k$, so is $\kappa_k$.

Corollary of Fact: If $X,Y$ are independent, then for all $k\ge1$,

$$\kappa_k(X+Y)=\kappa_k(X+Y,\dots,X+Y)=\kappa_k(X,\dots,X)+\kappa_k(Y,\dots,Y)=\kappa_k(X)+\kappa_k(Y).$$

Cumulants of $X+Y$ are the sums of those of $X$ and $Y$. Then also $\log \mathcal{E}_{X+Y}(t) = \log\mathcal{E}_X(t)+\log\mathcal{E}_Y(t)$: Cumulant-generating function linearizes independence.

Def: The <u>non-crossing cumulants</u> $\kappa_k:\mathcal{A}^k\to\mathbb{R}$, for $k\ge1$, are the multi-linear functions defined recursively by

$$\tau(a_1\dots a_k)=\sum_{\pi\in NC(k)}\prod_{S\in\pi}\kappa_{|S|}(a_S) \quad (*)$$

where $a_S=(a_i : i\in S)$ and $NC(k)$ is the set of non-crossing partitions of $\{1,\dots,k\}$. $(1\ 2\ 3\ 4\ 5\ 6)$

We write $\kappa_k(a)=\kappa_k(a,\dots,a)$. The <u>R-transform</u> of $a\in\mathcal{A}$ is

$$R_a(z)=\kappa_1(a)+\kappa_2(a)z+\kappa_3(a)z^2+\dots=\sum_{k\ge1}\kappa_k(a)z^{k-1}.$$

Prop: Define the Stieltjes transform of $a\in\mathcal{A}$ by

$$m_a(z) \text{ ``}=\tau((a-z)^{-1})\text{''} = -\frac1z-\frac{\tau(a)}{z^2}-\frac{\tau(a^2)}{z^3}-\dots = -\sum_{k\ge0} z^{-(k+1)}\tau(a^k).$$

Then $z=-\frac{1}{m_a(z)}+R_a(-m_a(z))$.

Proof: Check that the coefficient of each $\frac{1}{z^j}$ in $zm_a(z)+1$ matches that of $m_a(z)\cdot R_a(-m_a(z))$, using (*).

Note: When $|\tau(a^k)|\le C^k$ for all $k$, $R_a(z)$ and $m_a(z)$ are well-defined as analytic functions for small $|z|$ and $|z|^{-1}$, respectively, and $z=-\frac{1}{m_a(z)}+R_a(-m_a(z))$ holds as an equality of analytic functions for large $|z|$.

**Prop:** Let $a \in \mathcal{A}$ have semicircular law. Then $\mathcal{R}_a(z) = z$, and so

$$K_k(a) = \begin{cases} 1 & \text{for } k=2 \\ 0 & \text{for all other } k. \end{cases}$$

(The semicircle law is the free analogue of the standard Gaussian distribution.)

**Proof:** We know $1 + z m_a(z) + m_a(z)^2 = 0$, i.e. $z = -\frac{1}{m_a(z)} - m_a(z)$.

Then $\mathcal{R}_a(-m_a(z)) = -m_a(z)$ for all large $|z|$. Since both sides are non-constant and analytic in $z$, $\mathcal{R}_a(z) = z$ for all small $|z|$.

The statement about $K_k(a)$ follows from the series definition of $\mathcal{R}_a(z)$.

**Thm (Alternate Definition of Freeness):** Subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_m \in \mathcal{A}$ are free if and only if all of their mixed cumulants vanish. (I.e., for every $k \geq 2$ and $a_1, \dots, a_k \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_m$ where not all $a_1, \dots, a_k$ come from the same $\mathcal{A}_j$, $K_k(a_1, \dots, a_k) = 0$.)

**Corollary 1:** If $a, b \in \mathcal{A}$ are free, then $K_k(a+b) = K_k(a) + K_k(b)$ for each $k \geq 1$, and

$$\mathcal{R}_{a+b}(z) = \mathcal{R}_a(z) + \mathcal{R}_b(z).$$

**Proof:** This follows from multi-linearity of $K_k(a+b, \dots, a+b)$ and the above.

**Corollary 2:** If $a, b \in \mathcal{A}$ are free, with $|\tau(a^k)| \leq C^k$, $|\tau(b^k)| \leq C^k$, then

$$m_{a+b}(z) = m_a(w(z)) \text{ where } w(z) = z - \mathcal{R}_b(-m_{a+b}(z)).$$

**Remark:** This is a fixed-point equation in $m_{a+b}(z)$ involving only the individual laws of $a$ and $b$, via $m_a$ and $\mathcal{R}_b$. For $b$ semicircular, we have $\mathcal{R}_b(-m_{a+b}(z)) = -m_{a+b}(z)$. Then

$$m_{a+b}(z) = m_a(z + m_{a+b}(z))$$

which is the equation derived from integration by parts in Lecture 4.

When $|\tau(a^k)| \leq C^k$, there is a compactly supported measure $\mu$ s.t. $\tau(a^k) = \int x^k \, d\mu(x)$ for all $k$, and $m_a(z) = \int \frac{1}{x-z} \, d\mu(x)$, and the law of $a$ is identified with $\mu$. Similarly identifying $b$ with $\nu$, the law of $a+b$ is denoted

$$\mu \boxplus \nu,$$

the additive free convolution measure of $\mu$ and $\nu$. This is the limiting ESD of $X+Y$ when $X, Y \in \mathbb{R}^{n \times n}$ are asymptotically free w/ limiting ESDs $\mu$ and $\nu$.

**Proof of Corollary 2:** Define $f(w) = -w - \tau(a)w^2 - \tau(a^2)w^3 - \dots$, convergent and analytic for $|w| < C$. Then $f$ maps $\{w \in \mathbb{C} : |w| < C\}$ to an open neighborhood of $0$. So for each $z \in \mathbb{C}$ with $|z|$ sufficiently small, there exists $w$ with $|w| > \frac{1}{C}$ s.t. $z = m_a(w) = -\frac{1}{w} - \frac{\tau(a)}{w^2} - \frac{\tau(a^2)}{w^3} - \dots$

For sufficiently large $C_0 > 0$, define in this way a function $w(z)$ s.t.

$$m_{a+b}(z) = m_a(w(z))$$

for all $|z| > C_0$. Then for all such $z$,

$$\begin{aligned} w(z) &= -\frac{1}{m_a(w(z))} + \mathcal{R}_a(-m_a(w(z))) \\ &= -\frac{1}{m_{a+b}(z)} + \mathcal{R}_a(-m_{a+b}(z)) \\ &= -\frac{1}{m_{a+b}(z)} + \left[\mathcal{R}_{a+b}(-m_{a+b}(z)) - \mathcal{R}_b(-m_{a+b}(z))\right] \\ &= -\frac{1}{m_{a+b}(z)} + \left[z + \frac{1}{m_{a+b}(z)} - \mathcal{R}_b(-m_{a+b}(z))\right] \\ &= z - \mathcal{R}_b(-m_{a+b}(z)). \end{aligned}$$

Proof of Alternate Def. of Freeness: $\Leftarrow$: Suppose all mixed cumulants of $A_1, \dots, A_m$ vanish. If $a_1, \dots, a_k$ are s.t. $a_i \in A_{j(i)}$, $\tau(a_i) = 0$, and $j(1) \neq j(2), \dots, j(k-1) \neq j(k)$, then $\tau(a_1 \cdots a_k) = \sum_{\pi \in NC(k)} \prod_{S \in \pi} \kappa_{|S|}(a_S)$.

For each partition $\pi$, there is at least one set $S$ of consecutive elements, which either is of size 1 or come from $\geq 2$ distinct $A_1, \dots, A_m$. In either case $\kappa_{|S|}(a_S) = 0$ for this $S$, so $\tau(a_1 \cdots a_k) = 0$, and $A_1, \dots, A_m$ are free.

$\Rightarrow$: Lemma 1: For any $k \geq 2$ and $i \in \{1, \dots, k\}$,

$$\kappa_k(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_k) = 0.$$

Proof: Induct on $k$. For $k=2$, $\kappa_2(a, 1) = \tau(a \cdot 1) - \tau(a)\tau(1) = 0$.
Assuming this holds for $k-1$, write $a_i = 1$ and

$$\tau(a_1 \cdots a_{i-1} 1 a_{i+1} \cdots a_k) = \sum_{\pi \in NC(k)} \prod_{S \in \pi} \kappa_{|S|}(a_S)$$

$$= \kappa_k(a_1, \dots, a_k) + \sum_{\substack{\pi \in NC(k) \\ \{i\} \in \pi}} \prod_{S \in \pi} \kappa_{|S|}(a_S)$$

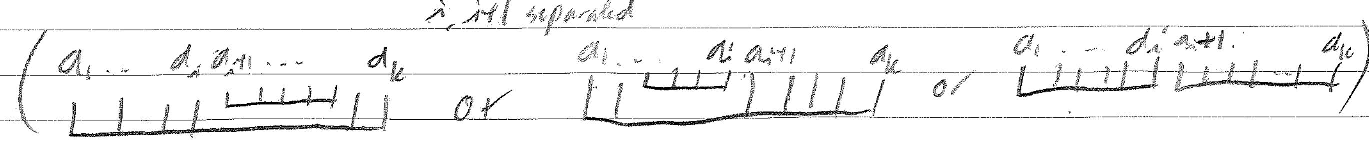
$$= \kappa_k(a_1, \dots, a_k) + \tau(a_1 \cdots a_{i-1} a_{i+1} \cdots a_k).$$

So $\kappa_k(a_1, \dots, a_k) = 0$.

Lemma 2: For any $k \geq 2$ and $i \in \{1, \dots, k-1\}$,

$$\kappa_{k-1}(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_k)$$

$$= \kappa_k(a_1, \dots, a_k) + \sum_{\substack{\pi \in NC(k) : |\pi| = 2 \\ i, i+1 \text{ separated}}} \prod_{S \in \pi} \kappa_{|S|}(a_S).$$




Proof: Induct on $k$. For $k=2$, $\kappa_1(a_1 a_2) = \tau(a_1 a_2) = \kappa_2(a_1, a_2) + \kappa_1(a_1)\kappa_1(a_2)$.
Assume this holds for $k-1$, write $(b_1, \dots, b_{k-1}) = (a_1, \dots, a_{i-1}, a_i a_{i+1}, \dots, a_k)$, and

$$\tau(a_1 \cdots a_k) = \sum_{\pi \in NC(k-1)} \prod_{S \in \pi} \kappa_{|S|}(b_S)$$

$$= \kappa_{k-1}(b_1, \dots, b_{k-1}) + \sum_{\substack{\pi \in NC(k-1) \\ |\pi| \geq 2}} \prod_{S \in \pi} \kappa_{|S|}(b_S)$$

Use induction hypothesis to expand the term containing $b_i$:

$$= \kappa_{k-1}(b_1, \dots, b_{k-1}) + \sum_{\substack{\pi \in NC(k) : \text{ some } \\ S \in \pi \text{ containing neither} \\ i \text{ nor } i+1,\ |\pi| \geq 2}} \prod_{S \in \pi} \kappa_{|S|}(a_S)$$

$$= \kappa_{k-1}(b_1, \dots, b_{k-1}) + \tau(a_1 \cdots a_k) - \kappa_k(a_1, \dots, a_k) - \sum_{\substack{\pi \in NC(k) : |\pi| = 2 \\ i, i+1 \text{ separated}}} \prod_{S \in \pi} \kappa_{|S|}(a_S).$$

Cancelling $\tau(a_1 \cdots a_k)$ and rearranging completes the induction.

Suppose $A_1, \dots, A_m$ free, $a_1, \dots, a_k$ not all from the same $A_j$. Need to show $\kappa_k(a_1, \dots, a_k) = 0$. By Lemma 1 and linearity, suffices to show this when $a_i$ are centered, i.e. $\tau(a_i) = 0$ for all $i$. Induct on $k$.

For $k=2$, $\kappa_2(a_1, a_2) = \tau(a_1 a_2) - \tau(a_1)\tau(a_2) = 0$ by freeness.

For $k>2$: If any two consecutive $a_i, a_{i+1}$ come from same $A_j$, use Lemma 2 to get

$$\kappa_k(a_1, \dots, a_k) = \kappa_{k-1}(a_1, \dots, a_{i-1}, a_i a_{i+1}, \dots, a_k) - \sum_{\substack{\pi \in NC(k) : |\pi| = 2 \\ i, i+1 \text{ separated}}} \prod_{S \in \pi} \kappa_{|S|}(a_S)$$

$= 0$ by induction hypothesis.

Otherwise, write

$$\kappa_k(a_1, \dots, a_k) = \tau(a_1 \cdots a_k) - \sum_{\substack{\pi \in NC(k) \\ |\pi| \geq 2}} \prod_{S \in \pi} \kappa_{|S|}(a_S)$$

0 by freeness

Some $S$ of size 1 or consists of consecutive elements, 0 by inductive hypothesis.

$= 0.$