Free Probability

Let $X \in \mathbb{R}^{n \times n}$ be Wigner, $A \in \mathbb{R}^{n \times n}$ symmetric, deterministic.

If $A$ has full rank $n$, eigenvalues of $X+A$ are semi-circle plus outliers.

If $A$ has full rank, the bulk eigenvalue distribution of $X+A$ changes. We saw this in Lecture 4. What about other matrices, e.g., $AXA$? $AXA + X$?

Example: Write $n = N + M$. Set

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Decompose the Wigner matrix $X$ as

$$X = \begin{pmatrix} X_1 & X_2 \\ X_2 & X_3 \end{pmatrix}$$

where $X_1$ is $X_1^T$, $X_3 = X_3^T$ (if $X\in \mathbb{R}^{n \times n}$ is Hermitian).

Then $A_1 X_1 A_1 X_1 A_1 = \begin{pmatrix} 0 & X_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_1 \\ 0 & 0 \end{pmatrix}$

has the same eigenvalues, plus $M$ 0's, as the sample covariance matrix $X_1$. $X_1$.

By understanding general polynomials of simple matrices, we can understand new complex matrix models.

Intuition: The eigenvalues of $X$ and $A$ are with high probability "aligned" or in "generic position". This is easiest to see when $X \sim \text{GOE}(n)$, but also true for general Wigner $X$.

One way to describe this: Let $\mu = \mathbb{E} \left[ A \right]$.

Suppose $P_1$, $P_2$ are polynomials s.t. $\mu P_1(X) \to 0$ for each fixed $A$.

Then $\mu P_1(X) = A \mu P_1(X) = A \mu P_1(X) \to 0$ as $n \to \infty$.

If $X, A$ satisfy this for all such $P_1, P_2$, $\mu$ and $\text{Fock}$, then they are called asymptotically free.

Calculation of moments: Write as shorthand $M = \mu \mu^T$.

Then $X^2 = \mu X + \mu A$.

Now $X^2 = \mu X + \mu A$.

$$\mathbb{E} \left[ (X^2)^2 \right] = \mathbb{E} \left[ \mu X + \mu A \right] \mathbb{E} \left[ \mu X + \mu A \right] = \mu X + \mu A$$

$$\implies \mathbb{E} \left[ (X^2)^2 \right] = \mu \mu X + \mu \mu A + \mu \mu A = \mu \mu X + \mu \mu A$$

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\[ z = x + A + X(A - A) + (z/A)^2 + 2z + x \]

This extends to an algorithm to compute \( \lim_{n \to \infty} P(X(A)) \) for any polynomial in \( X \) and \( A \), in terms of \( \lim_{n \to \infty} X^2 \) and \( \lim_{n \to \infty} x + A \).

Definition: Let \( A \) be a \( \mathbb{F} \)-algebra over \( \mathbb{C} \) with multiplicative unit 1. (So \( a = 1 = \{a, 0, a \} \)) (a, b), \( a^* = a, \ldots \) A function \( \tau: A \to \mathbb{C} \) is a trace if:

\[ \tau(x) = \tau(a) \]

\[ \tau(ab) = \tau(ba) \quad (so \ \tau(ab) = \tau(ba) = \tau(cab)) \]

\((A, \tau)\) is a noncommutative probability space.

Example: \( R = \mathbb{F} \) (random) matrices in \( \mathbb{C}^{n \times n} \). \( \tau = \tau = \mathbb{E}[X] \).

Definition: \( A_{n \times n} \) is a subalgebra if \( a, b \in A \) and \( \tau(ab) = \tau(ba) \).

The subalgebra generated by \( \{a_1, \ldots, a_k\} \) is all polynomials in \( \{a_1, \ldots, a_k\} \).

Definition: Subalgebras \( A_n, A_m \) are free if \( \tau(x \_1 \_2 \_m) = 0 \) whenever:

- Each \( \_i \) belongs to some \( A_{n_i} \) and \( \tau(x) = 0 \) and
- No two consecutive \( \_i \) belong to the same \( A_{n_i} \). I.e., \( j(1), \ldots, j(k) = j(1), \ldots, j(m) \).

Collecting all elements \( A_n, A_m \) are free if their generated subalgebra is.

In particular, \( a, \_k \) in the \( k^{th} \), \( a_1, a_2 \_1 \_2 \_m \) in the \( m^{th} \).

This is somewhat analogous to independence of collections of random variables.

Definition: Collections of (random) matrices \( A_{n \times n}, A_{m \times m} \) are asymptotically free with respect to \( \tau = \mathbb{E}[X] \) if there exist a map \( \tau: A \to \mathbb{C} \) and corresponding \( A_{n \times n}, A_{m \times m} \) such that for every fixed polynomial \( P \),

\[ \tau(P(A_{n \times n}, A_{m \times m}, \ldots)) = \tau(P(A_{n \times n}, A_{m \times m}, \ldots)) \]

as \( n \to \infty \) and \( A_{n \times n} \) are free in \( A_{n \times n} \).

Theorem: Let \( W_{1 \times 1}, W_{2 \times 2} \in \mathbb{C}^{n \times n} \) be independent Wigner matrices and \( D_{n \times n}, D_{m \times m} \) deterministic matrices such that \( \tau(\mathbb{E}[D_{n \times n} D_{m \times m}]) \) has a limit as \( n \to \infty \). For any fixed polynomial \( P \), \( \tau(\mathbb{E}[D_{n \times n} D_{m \times m}]) \) has a limit as \( n \to \infty \).

Then, if \( W_{1 \times 1}, W_{2 \times 2}, D_{n \times n} \) are deterministic matrices such that \( \tau(\mathbb{E}[D_{n \times n} D_{m \times m}]) \) has a limit as \( n \to \infty \), then exist \( W_{1 \times 1}, W_{2 \times 2} \) are asymptotically free. I.e., there exist \( W_{1 \times 1}, W_{2 \times 2} \) such that \( \tau(\mathbb{E}[W_{1 \times 1} W_{2 \times 2}]) \) is a trace.

For every fixed polynomial \( P \),

\[ \tau(\mathbb{E}[W_{1 \times 1} W_{2 \times 2} D_{n \times n} D_{m \times m}]) = \tau(\mathbb{E}[W_{1 \times 1} W_{2 \times 2} D_{n \times n} D_{m \times m}]) \]

Note: In particular, \( \tau(W_{1 \times 1}) = \tau(W_{2 \times 2}) \), so \( \tau(W_{1 \times 1}) \) must be the \( k^{th} \) moment of the semicircle law for every \( k \). This unique defines \( \tau(\mathbb{E}[w]) \) for every semicircle polynomial \( P \). The elements \( w_{1 \times 1}, W_{2 \times 2} \) are called free semicircular elements in \( A \).

Definition: The class of \( A_{n \times n} \) is called \( A_{n \times n} \) if \( \tau(\mathbb{E}[W_{1 \times 1} W_{2 \times 2}]) \) is the collection of moments.

\( \tau(w) \) is the moment in a semicircle. More generally, the (joint) law of \( A_{n \times n} \)

is the collection of joint moments \( \tau(w) \) is the moment in elements of \( A_{n \times n} \).
Prop: Suppose \(A_{11}, A_{12} \mid E \neq \emptyset\) are free. Then the law of \(A_{11}, A_{12} \mid E \neq \emptyset\) is uniquely determined by the individual laws of \(A_{11}, A_{12} \mid \emptyset\).

Proof: It suffices to show, for any \(a_{11}, a_{12}\) where each \(a_{ij}\) belongs to \(\mathbb{E}(A_{ij} \mid \emptyset)\), and \(j(1), j(2), j(3)\), \(j(1) \neq j(2) \neq j(3) \neq j(1)\), that \(\mathbb{E}(A_{11}, A_{12})\) is determined.

We induct on \(k\). For \(k=1\), \(\mathbb{E}(a_{1})\) is clearly determined by the law of \(A_{ij} \mid \emptyset\).

\[ A_{ij} \]

Suppose this holds up to \(k-1\). Write

\[ a_k = \mathbb{E}(a_k), \quad a_k = a_k + (a_k - \mathbb{E}(a_k)) \]

Then

\[ \mathbb{E}(a_1, a_k) = \mathbb{E} \left( \mathbb{E} \left( a_1, a_k \mid a_1, a_k \right) \right) = \mathbb{E} \left( a_1 - \mathbb{E}(a_1), a_k - \mathbb{E}(a_k) \right) + \mathbb{E} \text{(other terms)} \]

Each other term has at least one \(a_k\), which can be pulled out of \(\mathbb{E} \) as a scalar factor \(\mathbb{E}(a_k)\). The remaining term can be written as \(\mathbb{E}(a_1, a_k)\) for some \(k \leq \ell\), each \(a_k\) belonging to a single vector \(A_{ij}\), so it is determined by the induction hypothesis.

The first term \(\mathbb{E}(a_1, a_k) - \mathbb{E}(a_1, \mathbb{E}(a_k)) = 0\) by the definition of freeness. So \(\mathbb{E}(a_1, a_k)\) is also determined, completing the induction.

Using this induction gives an algorithm for computing \(\mathbb{E}(P(A_{11}, A_{12}))\) for any polynomial \(P\) in terms of the individual laws of \(A_{11}, A_{12}\).

Applying to independent Wigner \(W_{11}, W_{12}\) and deterministic \(D_{11}, D_{12}\), for any polynomial \(P\) we have

\[ \mathbb{E} \left( \frac{1}{n} \text{Tr} P(W_{11}, W_{12}, D_{11}, D_{12}) \right) = \mathbb{E} \left( \frac{1}{n} \text{Tr} P(D_{11}, D_{12}) \right) \]

and we may use this algorithm to compute \(\mathbb{E}(x_{11}, x_{12})\) in terms of moments of the semi-circular law and the limiting moments

\[ \mathbb{E}(x_{11}, x_{12}) = \lim_{n \to \infty} \frac{1}{n} \text{Tr} \mathcal{Q}(D_{11}, D_{12}) \]

of polynomials \(\mathcal{Q}\) of the deterministic matrices \(D_{11}, D_{12}\).

Free cumulants, R-transform, and additive free convolution

Q: Suppose \(a \mid \emptyset\) are free. How to compute \(\mathbb{E}(a_1 a_2)\) in a single way?

Analogy to classical probability: For a r.v. \(X\), its moments are \(\mathbb{E}(X^n)\).

We define a moment-generating function by

\[ \varphi(t) = \mathbb{E}(e^{tX}) = 1 + t \mathbb{E}(X) + \frac{t^2}{2} \mathbb{E}(X^2) + \frac{t^3}{6} \mathbb{E}(X^3) + \cdots \]

Related to the moments are a set of cumulants \(\kappa_k\):\n
\[ \kappa_1(X) = \mathbb{E}(X) \quad \text{(mean)} \]

\[ \kappa_2(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) \quad \text{(variance)} \]

\[ \kappa_3(X) = \mathbb{E}(X^3) - 3 \mathbb{E}(X) \mathbb{E}(X^2) + 2 \mathbb{E}^3(X) \]

\[ \kappa_4(X) = \mathbb{E}(X^4) - 4 \mathbb{E}(X) \mathbb{E}(X^3) + 6 \mathbb{E}(X)^2 \mathbb{E}(X^2) - 3 \mathbb{E}^4(X) \]

These are the coefficients of \(t\) in the cumulant-generating function

\[ \log \varphi(t) = \log \mathbb{E}(e^{tX}) = \sum_{k \geq 1} \frac{\kappa_k(X)}{k} t^k \]

Fact: \(\kappa_k(X) = \mathbb{E}(X_1 \cdots X_k) - \mathbb{E}(X_1) \cdots \mathbb{E}(X_k)\) (moment-cumulant relations)
Then \( K_k(x) = m_k(x) - \sum_{i=1}^{k-1} \sum_{j=1}^{i} T_i K_{ij}(x) \). (4)

The cumulants of the right are \( K_k(x_i) \), \( K_{kn}(x_i) \), of lower order.

So \( (x) \) may be used as an alternative recursive definition of \( K_k(x) \), in terms of moments \( m_k(x_1), m_k(x) \).

More generally, define the mixed moment \( m_k(x_1,...,x_k) = E[x_1,...,x_k] \), and the mixed cumulant via

\[
m_k(x_1,...,x_k) = \sum_{\pi \in \mathcal{P}(k)} \prod_{i=1}^{k} K_{\pi_i}(x_i)
\]

where \( X = (x_1,...,x_k) \), i.e.,

\[
K_k(x_1,...,x_k) = m_k(x_1,...,x_k) - \sum_{\pi \in \mathcal{P}(k)} \prod_{i=1}^{k} K_{\pi_i}(x_i)
\]

In particular, \( K_k(x_1,...,x_k) = m_k(x_1,...,x_k) - m_1(x_1)m_1(x_2) \).

We have \( m_k(x) = m_k(x_1,...,x_k) \), \( K_k(x) = K_k(x_1,...,x_k) \).

Fact: If \( X, Y \) are independent, then all mixed cumulants of \( X \) and \( Y \) other than \( K_k(x, y) \) and \( K_k(y, x) \) for \( k \geq 1 \) vanish.

E.g., \( K_2(x, y) = 0 \), \( K_3(x, y, z) = 0 \), \( K_4(x, y, z) = 0 \), etc.

Note that \( m_k \) is linear in each argument.

\[
E[ (a_1 X_1 + a_2 X_2)^k ] = \sum_{\pi \in \mathcal{P}(k)} \prod_{i=1}^{k} \sum_{j=1}^{k} a_{\pi_j} E[X_{\pi_j}^j]
\]

Then by (6) and induction on \( k \), so is \( K_k \).

Corollary of Fact: If \( X, Y \) are independent, then for all \( k \),

\[
K_k(x+y) = K_k(x, y) = K_k(x) + K_k(y)
\]

Cumulants of \( X + Y \) are the sums of those of \( X \) and \( Y \). Then also

\[
\log E_{(X+Y)}(e^x) = \log E_X(e^x) + \log E_Y(e^x)
\]

Cumulant generating function characterizes independence.

Def: The non-crossing cumulants \( K_k \), or \( K_k(x_1,...,x_k) \), for \( k \geq 1 \), are the multilinear functions defined recursively by

\[
\gamma(a_1,...,a_k) = \sum_{\pi \in \mathcal{P}(k)} \prod_{i=1}^{k} K_{\pi_i}(a_i)
\]

where \( a_i = (a_i, i \in S) \) and \( \mathcal{P}(k) \) is the set of non-crossing partitions of \( \{1,2,...,k\} \).

We write \( K_k(a) = K_k(a_1,...,a_k) \). The \( R \)-transform of \( a \) is

\[
R_k(a) = K_k(a) - K_k(a_1,...,a_k) + K_k(a) + \gamma(a) = \frac{a^k}{k!} - \sum_{k=0}^{\infty} \frac{\gamma(a)}{k!} a^k
\]

Prop: Define the Stieltjes transform of a set by

\[
m_{m}(a) = \frac{1}{2} \log \frac{1}{R_k(-m_{m}(a))}
\]

Then \( m_{m}(a) = \frac{1}{m_{m}(a)} + R_k(-m_{m}(a)) \).

Proof: Check that the coefficients of each \( \frac{1}{z} \) in \( zm_{m}(a) + 1 \) matches that of

\[
m_{m}(a) R_k(-m_{m}(a)) \text{ at } z = \frac{1}{a}.
\]

Note: When \( f_{\pi}(a) \) is \( C^k \) for all \( k \), \( R_k(a) \) and \( m_{m}(a) \) are well-defined as analytic functions for small \( \text{Re } a \) and \( |a| > 1 \), respectively, and

\[
\int = \frac{1}{m_{m}(a)} + R_k(-m_{m}(a)) \text{ holds as an equality of analytic functions for large } |a|.
\]
Prop: Let $a \in \mathbb{C}$ have semicircular law. Then $K_a(z) = z$, and so

$$K_a(z) = \begin{cases} 1 & \text{for } |z| < 2 \\ 0 & \text{for all other } z \end{cases}$$

(The semicircle law is the free analogue of the standard Gaussian distribution.)

Proof: We know $1 + 2\mathbb{E}X + \mathbb{E}X^2 = \mathbb{E}(X^2 - \mathbb{E}X^2)$. Then $K_a(\mathbb{E}X) = -\mathbb{E}X$ for all large $|z|$. Show both sides are non-constant and analytic in $z$. $K_a(\mathbb{E}X) = z$ for all small $|z|$. The statement about $K_a(z)$ follows from the series definition of $K_a(z)$.

Then (Alternate Definition of Freeness): Subalgebras $A_1, A_2, A_3$ are free if and only if all of their mixed cumulants vanish, (i.e., for any $k_2$ and $a_{i_{11}}, a_{i_{21}}, a_{i_{31}}$ there are not all $a_{i_{11}}, a_{i_{21}}, a_{i_{31}}$ come from the same $A_j$, $K_{(i_1, i_2, i_3)}(a)$).

Corollary: If $a, b \in A_1$ are free, then $K_{a+b}(z) = K_a(z) + K_b(z)$ for each $k_2$, and $K_{a+b}(z) = K_a(z) + K_b(z)$.

Proof: This follows from multiplicativity of $K_a(\mathbb{E}X, \mathbb{E}Y)$ and the above.

Corollary 2: If $a, b \in A_1$ are free with $|\mathbb{E}X| \leq C$, $|\mathbb{E}Y| \leq C$, then

$$\mathbb{E}(z^2) = \mathbb{E}(w(z))$$

where $w(z) = z - R_a(\mathbb{E}X)^2 + R_b(\mathbb{E}Y)^2$.

Remark: This is a first order equation in $\mathbb{E}(z^2)$ involving only the individual laws of $a$ and $b$ via $\mathbb{E}X$ and $\mathbb{E}Y$. For $b$ semicircular we have $R_b(\mathbb{E}X)^2 = -\mathbb{E}X^2$. Thus

$$\mathbb{E}(z^2) = \mathbb{E}(z + \mathbb{E}X)^2$$

which is the equation derived from integration by parts in Lecture 4.

When $|\mathbb{E}X| \leq C$: there's a compactly supported measure $\mu$ s.t.

$$\mu(a) = \int \mu \, d\lambda$$

for all $\lambda$, and $\mathbb{E}(X) = \int \mathbb{E}(X) \, d\lambda$. And the law of $a$ is identical with $\mu$. Similarly identifying $b$ with $\mu$ the law of $b$ with $\mu$. Denote $\mu(a)$.

The add, the free convolution measure of $\mu$ and $\mu$. This is the limiting ESD of $X+Y$ when $X, Y \in F$ are asymptotically free and thin.

Proof: Corollary 2: Define $\tau(w) = w - \mathbb{E}(w) - \mathbb{E}(w^2)$, $w \rightarrow \mathbb{E}w$, converges and analytic for $|w| < C$. Then $\lim_{|w| \rightarrow C} \tau(w) = 0$ in an open neighborhood of $0$. So for each $z \in \mathbb{C}$ with $|z| \rightarrow C$ sufficiently small, there exists $w$ with $|w| \leq C$ s.t. $z = \mathbb{E}(w)$.

For sufficiently large $C_0 > 0$, define in this way a function $\tau(w)$ s.t.

$$\mathbb{E}(w(w)) = \mathbb{E}(w(z))$$

for all $|w| > C_0$. Then for all such $z$,

$$w(z) = w(w(z)) + \tau(w(z))$$

$$= -\frac{1}{\mathbb{E}(w(z))} + R_a(-\mathbb{E}(w(z)))$$

$$= -\frac{1}{\mathbb{E}(w(z))} + \tau(-\mathbb{E}(w(z)))$$

$$= \frac{1}{\mathbb{E}(w(z))} \left[ R_a(-\mathbb{E}(w(z))) - \mathbb{E}(w(z)) \right]$$

$$= \frac{1}{\mathbb{E}(w(z))} \left[ \mathbb{E}(w(z)) - R_a(-\mathbb{E}(w(z))) \right]$$

$$= \mathbb{E}(w(z)) - R_a(-\mathbb{E}(w(z)))$$,

$$= \mathbb{E}(w(z)) - R_a(-\mathbb{E}(w(z))).$$
Proof 1: Alternative Definition of 

Suppose all mixed constants \( A_{i,j}, A_{m} \)

vanish. If \( a_{1}, a_{2}, \ldots, a_{n} \) are such \( a_{i} \in A_{i,j}, \forall a_{i} \neq 0 \) and \( j(1), j(2), \ldots, j(m) \) are distinct, then

\[
\tau(a_{1}, a_{2}) = \sum_{i=0}^{n} \frac{\Pi K_{i}(a_{i})}{s_{i+1}}
\]

For each partition \( n \), there exist 1 or more consecutive elements which either is 0 or 1 or more than \( n \) distinct 0, 1. In either case \( K_{i}(a_{i}) = 0 \) for this \( S \), so \( \tau(a_{1} - a_{2}) = 0 \) and \( a_{1}, a_{2} \) are free.

\( \Rightarrow \) Lemma 1: For any \( k \geq 1 \) and \( i, j, k \),

\[
K_{k}(a_{1}, a_{2}, a_{3}, \ldots, a_{k}) = 0.
\]

\( \Rightarrow \) Proof: Induct on \( k \). For \( k = 1 \), \( K_{1}(a_{1}) = \tau(a_{1}) - \tau(\alpha) \tau(a_{1}) = 0. \)

Assume this holds for \( k = 1 \). Let \( a_{1}, a_{2}, \ldots, a_{k} \) be such \( a_{l} \in A_{i,j}, \forall a_{l} \neq 0 \) and

\[
\tau(a_{1}, a_{2}, \ldots, a_{k}) = \sum_{i=0}^{n} \frac{\Pi K_{i}(a_{i})}{s_{i+1}}
\]

so

\[
K_{k}(a_{1}, a_{2}, \ldots, a_{k}) = K_{k}(a_{1}, a_{2}, \ldots, a_{k}) + \tau(a_{1}, a_{2}, \ldots, a_{k})
\]

\( \Rightarrow \) Lemma 2: For any \( k \geq 1 \) and \( i, j, k \),

\[
K_{k+1}(a_{1}, a_{2}, \ldots, a_{k}, a_{k+1}) = K_{k}(a_{1}, a_{2}, \ldots, a_{k}, a_{k+1}) + \tau(a_{1}, a_{2}, \ldots, a_{k}, a_{k+1})
\]

\( \Rightarrow \) Proof: Induct on \( k \). For \( k = 2 \), \( K_{2}(a_{1}, a_{2}) = \tau(a_{1}, a_{2}) = K_{2}(a_{1}, a_{2}) + K_{2}(a_{1}, a_{2}) \).

Assume this holds for \( k = 2 \). Let \( a_{1}, a_{2}, \ldots, a_{k} \) be such \( a_{l} \in A_{i,j}, \forall a_{l} \neq 0 \) and

\[
\tau(a_{1}, a_{2}, \ldots, a_{k}) = \sum_{i=0}^{n} \frac{\Pi K_{i}(a_{i})}{s_{i+1}}
\]

Cancelling \( \tau(a_{1}, a_{2}, \ldots, a_{k}) \) and rearranging completes the induction.

Suppose \( A_{i,j} \) are free, \( a_{1}, a_{2}, \ldots, a_{n} \) and all from the same \( X \). Need to show

\[
K_{k}(a_{1}, a_{2}, \ldots, a_{k}) = 0.
\]

By Lemma 1 and briefly, suffices to show this when

\( a_{1} \) is an element, i.e. \( \tau(a_{1}) = 0 \) for all \( a_{1} \). Induct on \( k \).

For \( k = 2 \), \( K_{2}(a_{1}, a_{2}) = \tau(a_{1}, a_{2}) = 0 \) by freeness.

For \( k > 2 \): If any two consecutive \( a_{1}, a_{2} \) come from some \( X \) we use Lemma 2 to get

\[
K_{k}(a_{1}, a_{2}, \ldots, a_{k}) = K_{k}(a_{1}, a_{2}, \ldots, a_{k}) + \sum_{i=0}^{n} \frac{\Pi K_{i}(a_{i})}{s_{i+1}}
\]

by inductive hypothesis.

Otherwise, write

\[
K_{k}(a_{1}, a_{2}, \ldots, a_{k}) = \sum_{i=0}^{n} \frac{\Pi K_{i}(a_{i})}{s_{i+1}}
\]

0 by freeness. Since \( \tau(a_{1}, a_{2}, \ldots, a_{k}) \) is a constant.

\( \Rightarrow \) by inductive hypothesis.

\[
= 0.
\]