Asymptotic Freeness of Random Matrices

Recall: Random matrices $A_1, \dots, A_k \in \mathbb{C}^{n \times n}$ are <u>asymptotically free</u> if there are $a_1, \dots, a_k \in \mathcal{A}$, $\mathcal{A}$ a $*$-algebra with trace τ, s.t. $a_1, \dots, a_k \in \mathcal{A}$ are <u>free</u> and

$$\frac{1}{n} \mathbb{E}\, \mathrm{Tr}\, P(A_1, \dots, A_k, A_1^*, \dots, A_k^*) \to \tau(P(a_1, \dots, a_k, a_1^*, \dots, a_k^*))$$

for any fixed polynomial P in $2k$ variables.

Collections of matrices $\mathcal{A}_1 = \{A_1^{(1)}, \dots, A_k^{(1)}\}$ and $\mathcal{A}_2 = \{A_1^{(2)}, \dots, A_\ell^{(2)}\}$ are asymptotically free if there are $a_1^{(1)}, \dots, a_k^{(1)}, a_1^{(2)}, \dots, a_\ell^{(2)} \in \mathcal{A}$ s.t. $\{a_1^{(1)}, \dots, a_k^{(1)}\}$ is free of $\{a_1^{(2)}, \dots, a_\ell^{(2)}\}$ and for any polynomial P

$$\frac{1}{n} \mathbb{E}\, \mathrm{Tr}\, P(A_1^{(1)}, \dots, A_\ell^{(2)}, A_1^{(1)*}, \dots, A_\ell^{(2)*}) \to \tau(P(a_1^{(1)}, \dots, a_\ell^{(2)*})).$$

$a_1, \dots, a_k \in \mathcal{A}$ are <u>free</u> if one of these two equivalent conditions holds:

- $\tau(p_1 \cdots p_m) = 0$ whenever $m \geq 1$, each p_i is a polynomial of a single $a_{j(i)}$ where $\tau(p_i) = 0$, and $j(1) \neq j(2), j(2) \neq j(3), \dots, j(m-1) \neq j(m)$.
- $K_\ell(p_1, \dots, p_\ell) = 0$ whenever $\ell \geq 2$, each p_i is a polynomial of a single $a_{j(i)}$, and not all $j(i)$'s are identical.

Here K_ℓ is the non-crossing cumulant defined recursively by

$$\tau(x_1 \cdots x_m) = \sum_{\pi \in NC(m)} \prod_{S \in \pi} K_{|S|}(x_i : i \in S).$$

Similarly for collections of elements of $\mathcal{A}$.

Thm: Let $W_1, \dots, W_k \in \mathbb{R}^{n \times n}$ be independent Wigner matrices, $D_1, \dots, D_\ell \in \mathbb{C}^{n \times n}$ deterministic matrices which have a joint limit law:

$$\frac{1}{n} \mathrm{Tr}\, P(D_1, \dots, D_\ell, D_1^*, \dots, D_\ell^*) \text{ converges for any fixed polynomial } P.$$

Then (for fixed m, ℓ, as $n \to \infty$) $\{W_1\}, \{W_2\}, \dots, \{W_k\}, \{D_1, D_2, \dots, D_\ell\}$ are asymptotically free.

We'll show this in two cases:

① No deterministic matrices $D_1, \dots, D_\ell$.

② $D_1, \dots, D_\ell$ diagonal, $\|D_1\|, \dots, \|D_\ell\| \leq C$ for all n.

Thm: Let $W_1, \dots, W_k \in \mathbb{R}^{n \times n}$ be independent Wigner. Then as $n \to \infty$, they are asymptotically free.

Proof: Let $w_1, \dots, w_k \in \mathcal{A}$ be free semi-circular elements in some $*$-algebra $\mathcal{A}$. (We'll assume these exist.) This means:

- $w_1, \dots, w_k$ are free
- For each $\ell \geq 1$, $\tau(w_i^\ell) = \int x^\ell \, d\mu(x)$ where μ is the semi-circle law.

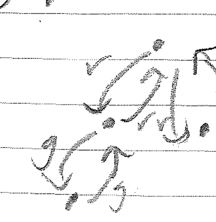
  Equivalently, for each $\ell \geq 1$, $K_\ell(w_i) = \begin{cases} 1 & \text{if } \ell = 2 \\ 0 & \text{otherwise.} \end{cases}$

It suffices to show $\frac{1}{n} \mathbb{E}\, \mathrm{Tr}\, P(W_1, \dots, W_k) \to \tau(P(w_1, \dots, w_k))$ for any monomial P. I.e., fix the degree m of P, and $\sigma(1), \dots, \sigma(m) \in \{1, \dots, k\}$. Then we must show

$$\frac{1}{n} \mathbb{E}\, \mathrm{Tr}\, W_{\sigma(1)} \cdots W_{\sigma(m)} \to \tau(w_{\sigma(1)} \cdots w_{\sigma(m)}).$$

Recall $$\frac{1}{n} \mathbb{E}\, \mathrm{Tr}\, W_{\sigma(1)} \cdots W_{\sigma(m)} = \frac{1}{n} \sum_{\substack{(i_1, \dots, i_m) \\ \text{paths on } \{1, \dots, n\}}} \mathbb{E}[W_{\sigma(1)}[i_1, i_2] W_{\sigma(2)}[i_2, i_3] \cdots W_{\sigma(m)}[i_m, i_1]].$$

Exactly as in the proof of the semi-circle law, paths which visit $< \frac{m}{2} + 1$ distinct vertices do not contribute asymptotically to the sum. The remaining paths correspond to Dyck words:

 $\Longleftrightarrow$ $\underset{r}{(}\,\underset{g}{(}\,\underset{g}{)}\,\underset{r}{)}\,\underset{r}{(}\,\underset{r}{)}$

However, each step $i \in \{1, \dots, m\}$ is now decorated with a "color" $\sigma(i) \in \{1, \dots, k\}$.

The expectation vanishes unless matching ( ) have the same color.

So $\frac{1}{n}\mathbb{E}\,\mathrm{Tr}\,W_{\sigma(1)}\cdots W_{\sigma(m)} \to$ # Dyck words "compatible" with the given sequence of colors $\sigma(1),\dots,\sigma(m)$.

Note: If all colors are the same, this is the Catalan number $C_{m/2}$ as before.

More generally, we claim this is exactly $\tau(w_{\sigma(1)}\cdots w_{\sigma(m)})$.

Why? $\tau(w_{\sigma(1)}\cdots w_{\sigma(m)}) = \sum_{\pi\in NC(m)} \prod_{S\in\pi} \kappa_{|S|}(w_{\sigma(i)} : i\in S)$.

Note that
$$\kappa_{|S|}(w_{\sigma(i)} : i\in S) = \begin{cases} 0 & \text{if } \sigma(i) : i\in S \text{ are not all the same} \\ 0 & \text{if } \sigma(i) : i\in S \text{ are all the same, but } |S|\neq 2 \\ 1 & \text{otherwise.} \end{cases}$$

So $\tau(w_{\sigma(1)}\cdots w_{\sigma(m)}) = \#\{$non-crossing <u>pairings</u> in $NC(m)$ which pair elements of the same color$\}$. This is exactly the above.

red green green red red red

1 2 3 4 5 6 $\iff$ (( )) ( ) with colors r g g r r r.

Thm: Let $W_1,\dots,W_k \in \mathbb{R}^{n\times n}$ be Wigner, $D_1,\dots,D_\ell \in \mathbb{C}^{n\times n}$ diagonal, bounded norm, w/ a limit joint law w.r.t. $\frac{1}{n}\mathrm{Tr}$. Then $W_1,\dots,W_k,\{D_1,\dots,D_\ell\}$ are asymptotically free.

Proof: Let $w_1,\dots,w_k,d_1,\dots,d_\ell\in\mathcal{A}$ be s.t. $w_1,\dots,w_k,\{d_1,\dots,d_\ell\}$ are free, each w_i is semi-circular, and for any polynomial P
$$\frac{1}{n}\mathrm{Tr}\,P(D_1,\dots,D_\ell,D_1^*,\dots,D_\ell^*) \to \tau(P(d_1,\dots,d_\ell,d_1^*,\dots,d_\ell^*)).$$
(Again, we'll assume these exist.) We must show convergence of $\frac{1}{n}\mathbb{E}\,\mathrm{Tr}$ of any monomial in W_i's, D_i's, and D_i^*'s. Without loss of generality, we may consider a monomial of the form, for $m\geq 1$ and $\sigma(1),\dots,\sigma(m)\in\{1,\dots,k\}$,
$$D_1 W_{\sigma(1)} D_1 W_{\sigma(2)} D_2 \cdots W_{\sigma(m)}$$

by
- relabeling products of D_i's and their transposes as $D_1,\dots,D_m$ as necessary, and doing the same for d_i's, and
- introducing $D_i \doteq I$ between consecutive W_i terms, as necessary.

So we want to show
$$\frac{1}{n}\mathbb{E}\,\mathrm{Tr}\,D_1W_{\sigma(1)}\cdots D_mW_{\sigma(m)} \to \tau(d_1w_{\sigma(1)}\cdots d_mw_{\sigma(m)}).$$

First look at
$$\tau(d_1w_{\sigma(1)}\cdots d_mw_{\sigma(m)}) = \sum_{\tilde\pi\in NC(2m)}\prod_{S\in\tilde\pi}\kappa_{|S|}(\text{some mix of } w\text{'s and } d\text{'s}).$$

The term for $\tilde\pi$ is 0 unless:

- m is even
- $\tilde\pi$ restricted to $w_{\sigma(1)},\dots,w_{\sigma(m)}$ is a non-crossing pairing of elements of the same "color"

Label $[2m]$ as $\{1',1,2',2,\dots,m',m\}$, where $1,\dots,m$ are colored by $\sigma(1),\dots,\sigma(m)$:

E.g.: $1'$ 1 $2'$ 2 $3'$ 3 $4'$ 4 $5'$ 5 $6'$ 6, with 1, 2, 3, 4, 5, 6 colored r g g r r r.

To sum over such $\tilde\pi$, we may

- first sum over non-crossing pairings π of $1,\dots,m$
- then sum over non-crossing partitions π' of $1',\dots,m'$ which do not cross π.

Def: Given $\pi\in NC(m)$ any non-crossing partition of $1,\dots,m$, its <u>Kreweras complement</u> $K(\pi)\in NC(m)$ is the coarsest non-crossing partition of $1',\dots,m'$ s.t. π & $K(\pi)$ combined yield a non-crossing partition of $1',1,2',2,\dots,m',m$.

Fixing π, the second sum over π' is over all refinements (further NC-partitions) of $K(\pi)$. We may write

$$\sum_{\pi' \text{ refines } K(\pi)} \prod_{S \in \pi'} \kappa_{|S|}(d_i : i \in S)$$

$$= \prod_{S \in K(\pi)} \underbrace{\sum_{\pi_S \text{ NC-partitions of } S} \prod_{T \in \pi_S} \kappa_{|T|}(d_i : i \in T)}_{= \tau(d_i : i \in T)}$$

$$= \prod_{S \in K(\pi)} \tau(d_i : i \in S).$$

Thus: $\tau(d_1 w_{\sigma(1)} \ldots d_m w_{\sigma(m)})$

$$= \sum_{\substack{\text{colored NC-pairings} \\ \pi \text{ of } \{1,..,m\}}} \prod_{S \in K(\pi)} \tau(d_i : i \in S).$$

Now look at

$$\frac{1}{n} \mathbb{E} \operatorname{Tr} D_1 W_{\sigma(1)} \ldots D_m W_{\sigma(m)}$$

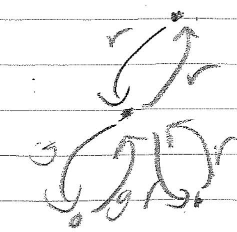
$$= \frac{1}{n} \sum_{i_1,..,i_m} \mathbb{E}\Big[D_1[i_1,i_1] W_{\sigma(1)}[i_1,i_2] D_2[i_2,i_2] W_{\sigma(2)}[i_2,i_3] \ldots D_m[i_m,i_m] W_{\sigma(m)}[i_m,i_1]\Big]$$

The term for path $(i_1,..,i_m)$ is

- 0 if some edge (i,j) appears exactly once for any unique color
- Applying $|D[i,i]| \leq C$, negligible if # distinct vertices is $< \frac{m}{2}+1$.

The non-negligible paths correspond to colored NC-pairings of $1,..,m$.

Take one such path, e.g.



$\Longleftrightarrow$ $1' \; 1 \; 2' \; 2 \; 3' \; 3 \; 4' \; 4 \; 5' \; 5 \; 6' \; 6$

This path means: $i_2 = i_4 = i_6$. Then the contribution to the expected trace is

$$\frac{1}{n^{6/2+1}} \sum_{\substack{i_1,i_2,i_3,i_5 \\ \text{distinct}}} D_1[i_1,i_1] D_2[i_2,i_2] D_3[i_3,i_3] D_4[i_2,i_2] D_5[i_5,i_5] D_6[i_2,i_2]$$

$$= \left(\frac{1}{n} \operatorname{Tr} D_1\right) \cdot \left(\frac{1}{n} \operatorname{Tr} D_2 D_4 D_6\right) \left(\frac{1}{n} \operatorname{Tr} D_3\right) \left(\frac{1}{n} \operatorname{Tr} D_5\right) + o(1)$$

$$\to \prod_{S \in K(\pi)} \tau(d_i : i \in S), \text{ where } \pi \text{ is the NC-pairing for this path.}$$

In general, for a pairing π, if a is paired with b, then

$$(i_a, i_{a+1}) = (i_{b+1}, i_b)$$

which enforces $i_a = i_{b+1}$, $i_{a+1} = i_b$. Can check that this always induces the Kreweras complement $K(\pi)$ on $1',..,m'$. So

$$\frac{1}{n} \mathbb{E} \operatorname{Tr} D_1 W_{\sigma(1)} \ldots D_m W_{\sigma(m)} \to \sum_{\substack{\text{colored NC-pairings} \\ \pi \text{ of } \{1,..,m\}}} \prod_{S \in K(\pi)} \tau(d_i : i \in S)$$

$$= \tau(d_1 w_{\sigma(1)} \ldots d_m w_{\sigma(m)}).$$

<u>Sample covariance matrices, Marcenko-Pastur law</u>

Def: Let $Y \in \mathbb{R}^{n \times p}$ have independent rows with mean 0, covariance $\Sigma \in \mathbb{R}^{p \times p}$.

Assume this arises as $Y = X \Sigma^{1/2}$, where $X \in \mathbb{R}^{n \times p}$ has independent entries,

$$\mathbb{E}[X_{ij}] = 0, \quad \mathbb{E}[X_{ij}^2] = 1, \quad \mathbb{E}[|X_{ij}|^k] \leq C_k \text{ for all } k \geq 0.$$

We'll call $\hat{\Sigma} = \frac{1}{n} Y^T Y = \frac{1}{n} \Sigma^{1/2} X^T X \Sigma^{1/2}$ the <u>sample covariance matrix</u>.

of Y. Note that $\hat{\Sigma}_{ij} = \frac{1}{n} Y_i^T Y_j$ where $Y_1, \ldots, Y_p$ are the columns of Y: This is the sample covariance (without mean-centering) between Y_i and Y_j.

If $\Sigma = I$, we'll call $\hat{\Sigma}$ a <u>white</u> sample covariance.

Thm (Marcenko, Pastur '67): Suppose $\Sigma = I$ and $n, p \to \infty$ s.t. $\frac{p}{n} \to \gamma \in (0, \infty)$. Then the ESD of $\hat{\Sigma}$ converges weakly a.s. to a limit μ_γ, called the <u>Marcenko-Pastur distribution</u> (with parameter γ).

Characterizations/properties:

① For each $z \in \mathbb{C}^+$, the equation

$$\gamma z m(z)^2 + (z + \gamma - 1) m(z) + 1 = 0$$

has a unique root $m(z) \equiv m_\gamma(z) \in \mathbb{C}^+$. μ_γ is the distribution with Stieltjes transform m_γ.

② Solving gives

$$m_\gamma(z) \in \frac{-(z+\gamma-1) \pm \sqrt{(z+\gamma-1)^2 - 4\gamma z}}{2\gamma z}$$

$$= \frac{-(z+\gamma-1) \pm \sqrt{(\gamma_+ - z)(\gamma_- - z)}}{2\gamma z}, \quad \gamma_\pm = (1 \pm \sqrt{\gamma})^2$$

Then

$$\lim_{\eta \downarrow 0} \frac{1}{\pi} \operatorname{Im} m_\gamma(x + i\eta) = \begin{cases} \frac{1}{2\pi\gamma x} \sqrt{(\gamma_+ - x)(x - \gamma_-)} & \text{if } x \in [\gamma_-, \gamma_+]. \\ 0 & \text{if } x \notin [\gamma_-, \gamma_+] \text{ and } x \neq 0. \end{cases}$$

Thus at all $x \neq 0$, μ_γ has the <u>density</u>

$$d\mu_\gamma(x) = \frac{1}{2\pi\gamma x} \sqrt{(\gamma_+ - x)(x - \gamma_-)} \, \mathbb{1}\{x \in [\gamma_-, \gamma_+]\}.$$

If $\gamma \leq 1$, this integrates to 1. If $\gamma > 1$, this integrates to $\frac{1}{\gamma}$, and μ_γ has a point mass of $1 - \frac{1}{\gamma}$ at 0.

③ Rearranging gives

$$z = -\frac{1}{m_\gamma(z)} + \frac{1}{1 + \gamma m_\gamma(z)}.$$

Recall the R-transform satisfies $z = -\frac{1}{m(z)} + R(-m(z))$.

Comparing, we see that μ_γ has the R-transform

$$R_\gamma(z) = \frac{1}{1 - \gamma z} = \sum_{k=1}^{\infty} \gamma^{k-1} z^{k-1},$$

and non-crossing cumulants $\kappa_k = \gamma^{k-1}$ for each $k \geq 1$. (Mean 1, variance γ.)

④ The moments of μ_γ are then

$$\int x^k d\mu_\gamma(x) = \sum_{\pi \in NC(k)} \prod_{S \in \pi} \kappa_{|S|} = \sum_{\pi \in NC(k)} \gamma^{k - |\pi|},$$

where $|\pi|$ is the number of sets in the partition π.

Check: Number of $\pi \in NC(k)$ with $|\pi| = t$ is $\frac{1}{k}\binom{k}{t}\binom{k}{t-1}$. So

$$\int x^k d\mu_\gamma(x) = \sum_{t=1}^{k} \gamma^{k-t} \cdot \frac{1}{k}\binom{k}{t}\binom{k}{t-1}.$$

Proof of M-P Thm: We'll use the moment method, and show

$$\frac{1}{p} \mathbb{E} \operatorname{Tr} \hat{\Sigma}^k \longrightarrow \int x^k d\mu_\gamma(x) = \sum_{\pi \in NC(k)} \gamma^{k - |\pi|}.$$

A concentration of measure argument concludes a.s. convergence. Let $N = n + p$. Embed X in a Wigner matrix

$$W = \frac{1}{\sqrt{N}} \begin{array}{c} \begin{pmatrix} W_1 & X^T \\ X & W_2 \end{pmatrix} \\ \begin{matrix} p & \quad n \end{matrix} \end{array} \in \mathbb{R}^{N \times N}. \quad \text{Let } P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Then $P_1 W P_2 W P_1 = \frac{1}{N} \begin{pmatrix} X^T X & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{n}{N} \hat{\Sigma} & 0 \\ 0 & 0 \end{pmatrix}$.

So $\frac{1}{p}\mathbb{E}\,\mathrm{Tr}\,\hat{\Sigma}^k = \frac{N}{p}\left(\frac{N}{n}\right)^k \cdot \frac{1}{N}\mathbb{E}\,\mathrm{Tr}\,(P_1 W P_2 W P_1)^k$.

Let $w, p_1, p_2 \in \mathcal{A}$ s.t.

① w is free of $\{p_1, p_2\}$ and semi-circular.

② $\tau(p_1) = \lim_{n,p\to\infty} \frac{1}{N}\mathrm{Tr}\,P_1 = \frac{\gamma}{1+\gamma}$, $\tau(p_2) = \lim_{n,p\to\infty} \frac{1}{N}\mathrm{Tr}\,P_2 = \frac{1}{1+\gamma}$.

③ $p_1^2 = p_1$, $p_2^2 = p_2$, $p_1 p_2 = 0$.

(We'll assume such a space exists.) ② and ③ imply $\frac{1}{N}\mathrm{Tr}\,Q(P_1, P_2) \to \tau(Q(p_1,p_2))$ for any polynomial Q.

Then, since $\frac{p}{N} \to \frac{\gamma}{1+\gamma}$, $\frac{n}{N} \to \frac{1}{1+\gamma}$, we get from asymptotic freeness

$$\frac{1}{p}\mathbb{E}\,\mathrm{Tr}\,\hat{\Sigma}^k \to \frac{1+\gamma}{\gamma}\cdot(1+\gamma)^k \cdot \tau\left((p_1 w p_2 w p_1)^k\right).$$

We have

$$\tau\left((p_1 w p_2 w p_1)^k\right) = \tau\Big(\underbrace{p_1 w p_2 w \; p_1 w p_2 w \; \cdots \; p_1 w p_2 w}_{k \text{ times}}\Big)$$

$$= \sum_{\substack{\pi \in NC\text{-pairings} \\ \text{of } \{1,\dots,2k\}}} \prod_{S \in K(\pi)} \tau(p_i : i \in S).$$

E.g. for $k=4$, one such term is

$$p_1 \; w \; p_2 \; w \; p_1 \; w \; p_2 \; w \; p_1 \; w \; p_2 \; w \; p_1 \; w \; p_2 \; w$$

$$\tau(p_1)\cdot\tau(p_2^3)\,\tau(p_1^2)\,\tau(p_2)\,\tau(p_1) = \tau(p_1)^3\tau(p_2)^2 = \left(\frac{\gamma}{1+\gamma}\right)^3\left(\frac{1}{1+\gamma}\right)^2.$$

Observe:

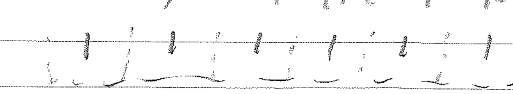
- Each w preceded by p_1, followed by p_2 must be paired w/ w preceded by p_2, followed by p_1. Otherwise, there are an odd number of w's inside this pair, which can't be paired w/ each other.
- For each pair of w's, the p_i following the first w and the p_i preceding the second w belong to the same element of the Kreweras complement $K(\pi)$. Similarly, the p_j preceding the first w and p_j following second w belong to same element of $K(\pi)$. These requirements completely define $K(\pi)$.
- In particular, given the pairing π, the elements of $K(\pi)$ containing p_2 form a non-crossing partition π_2 of the k p_2's, those containing p_1 form a non-crossing partition π_1 of the k p_1's, and these two partitions are Kreweras complements of each other.
- Given the partition π_2 of p_2's, we can recover π_1 by its Kreweras complement (in $NC(k)$). We can then recover the pairing of w's, π, by the Kreweras complement of $\pi_1 \cup \pi_2$ in $NC(2k)$.

So there is a bijection $\pi \longleftrightarrow \pi_2$.

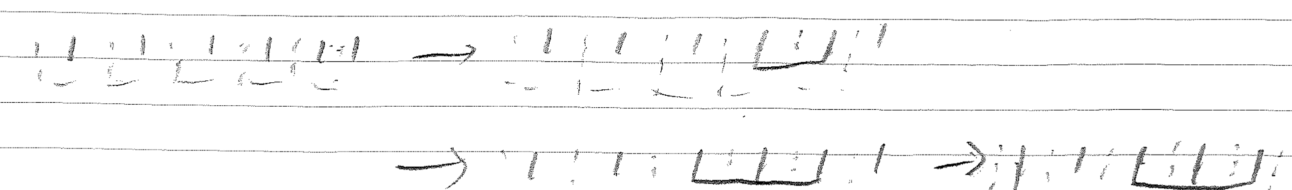
Then $$\tau\left((p_1 w p_2 w p_1)^k\right) = \sum_{\pi_2 \in NC(k)} \tau(p_2)^{|\pi_2|}\cdot\tau(p_1)^{|K(\pi_2)|}$$ $|\pi_1|$

$$= \sum_{\pi_2 \in NC(k)} \left(\frac{1}{1+\gamma}\right)^{|\pi_2|}\cdot\left(\frac{\gamma}{1+\gamma}\right)^{|K(\pi_2)|}$$

Claim: For any $\pi \in NC(k)$, $|\pi| + |K(\pi)| = k+1$.

Start with 

So $|\pi| = k$, $|K(\pi)| = 1$. To reach any π, we may iteratively connect two elements of π:



Each time $|\pi|$ decreases by 1, $|K(\pi)|$ increases by 1.

So $|\pi| + |K(\pi)| = k+1$ always.

Thus $\tau\left((p_1 w p_2 w p_1)^k\right) = \sum_{\pi_2 \in NC(k)} \left(\frac{1}{1+\gamma}\right)^{|\pi_2|} \left(\frac{\gamma}{1+\gamma}\right)^{k+1-|\pi_2|}$

$$= \sum_{\pi \in NC(k)} \left(\frac{1}{1+\gamma}\right)^{k+1} \gamma^{k+1-|\pi|}.$$

Then $\frac{1}{p} \mathbb{E} \operatorname{Tr} \hat{\Sigma}^k \to \frac{1+\gamma}{\gamma} (1+\gamma)^k \cdot \sum_{\pi \in NC(k)} \left(\frac{1}{1+\gamma}\right)^{k+1} \gamma^{k+1-|\pi|}$

$$= \sum_{\pi \in NC(k)} \gamma^{k-|\pi|},$$ as desired.

<u>Problem 6.</u> Let $W_1, \dots, W_k \in \mathbb{R}^{n \times n}$ be independent Wigner matrices, and $v, w \in \mathbb{R}^n$ deterministic vectors s.t. $v^T w \to \rho \in [-1,1]$, as $n \to \infty$. Let $w_1, \dots, w_k \in \mathcal{A}$ be free semi-circular elements. Show, for any fixed polynomial $P(x_1, \dots, x_k)$, that $\mathbb{E}\, v^T P(W_1, \dots, W_k) w \to \rho \cdot \tau(P(w_1, \dots, w_k))$.