Spectra of sparse random graphs

Let $G(n, \frac{d}{n})$ be the Erdős-Rényi random graph on n vertices: Independently for each pair $i<j$, $\mathbb{P}[i \sim j] = \frac{d}{n}$.

Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix:

$$A_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

Then $X_{ij} = \frac{1}{\sqrt{n \cdot \frac{d}{n}(1-\frac{d}{n})}} \left(A_{ij} - \frac{d}{n} \mathbb{1}\{i \neq j\} \right)$ defines a Wigner matrix:

- $\mathbb{E}[X_{ij}] = 0$, $\mathbb{E}[(\sqrt{n} \cdot X_{ij})^2] = 1$
- $\mathbb{E}[|\sqrt{n} X_{ij}|^k] = \left[\frac{d}{n} \cdot \left(\frac{1-d/n}{\sqrt{d/n \cdot (1-d/n)}} \right)^k + \left(1-\frac{d}{n}\right) \left(\frac{d/n}{\sqrt{d/n \cdot (1-d/n)}} \right)^k \right]$

$$\sim (n/d)^{k/2-1} \quad \text{if } d = d_n \ll n.$$

Previously, we assumed $\mathbb{E}[|\sqrt{n} X_{ij}|^k] \leq C_k$ for a constant $C_k > 0$ independent of n.

Using the types of techniques we've covered, one can show

- If $d = d_n \gg 1$ (i.e. $d \to \infty$ at any rate with n) then the ESD of X converges (weakly in probability) to the semicircle law.

  See Tran, Vu, Wang "Sparse random graphs," 2013.

- If $d \gg \log n$, then $\|X\| \to 2$ in probability.

  See Benaych-Georges, Bordenave, Knowles "Spectral radii of sparse random matrices," 2018.

Q: What happens if d is constant in n? This is called the <u>dilute</u> regime.

- Consider $\mathbb{E}\left[\frac{1}{n}\operatorname{Tr} X^k\right] = \frac{1}{n}\sum_{i_1,i_2,\ldots,i_k} \mathbb{E}[X_{i_1 i_2} \cdots X_{i_k i_1}]$.

  Consider even k, and the path $i_1, i_2, i_1, i_2, \ldots, i_1, i_2$.

  The contribution from this path is

  $$\frac{1}{n}\cdot n(n-1)\cdot \mathbb{E}[X_{ij}^k] \approx \frac{1}{n}\cdot n(n-1)\cdot \frac{d}{n}\left(\frac{1}{\sqrt{d}}\right)^k = d^{1-k/2}.$$

  Does <u>not</u> vanish as $n\to\infty$, even though this visits only two vertices.

- Consider $R(z) = (X - zI)^{-1}$. We have

  $$R_{ii}(z) = \frac{1}{-z - x_i^T R^{(i)}(z) x_i}.$$

  The quantity $x_i^T M x_i \approx \frac{1}{d}(a_i - \frac{d}{n}1)^T M (a_i - \frac{d}{n}1)$

  does <u>not</u> concentrate around $\mathbb{E}[x_i^T M x_i]$ as $n\to\infty$:

  E.g. for $M = I$, $\mathbb{E}[x_i^T M x_i] = \sum_j \mathbb{E}[x_{ij}^2] = 1$, and

  $$\operatorname{Var}[x_i^T M x_i] = \sum_j \mathbb{E}\left[\left(x_{ij}^2 - \frac{1}{n}\right)^2\right] = \sum_j \left[(\mathbb{E} X_{ij}^4) - \frac{1}{n^2}\right] \approx \sum_j \frac{1}{dn} = \frac{1}{d}.$$

  Individual entries of the resolvent remain random in the limit $n\to\infty$.

Indeed: The ESD of A (and of X) converges to a limit distribution w/ unbounded support, which is <u>not</u> the semicircle law.

One implication: Consider a stochastic block model $G(n, \frac{a}{n}, \frac{b}{n})$ where

$$\mathbb{P}[i\sim j] = \begin{cases} \frac{a}{n} & \text{if } i,j \le \frac{n}{2} \text{ or } i,j \ge \frac{n}{2}+1 \\ \frac{b}{n} & \text{if } i \le \frac{n}{2}, j \ge \frac{n}{2}+1 \text{ or } i \ge \frac{n}{2}+1, j \le \frac{n}{2}. \end{cases}$$

$$\mathbb{E}[A] = \begin{array}{c} \begin{array}{cc} n/2 & n/2 \end{array} \\ \left(\begin{array}{c|c} \frac{a}{n} & \frac{b}{n} \\ \hline \frac{b}{n} & \frac{a}{n} \end{array}\right) \end{array}$$ is rank two, with eigenvalues $\frac{a+b}{2}$ and $\frac{a-b}{2}$.

For a dilute graph with a,b constant in n, the "signal eigenvalue" $\frac{a-b}{2}$ is covered by the bulk of the spectrum as $n\to\infty$. Applying a spectral decomposition of A fails to uncover the community structure.

<u>Graph spectra and local weak convergence</u>

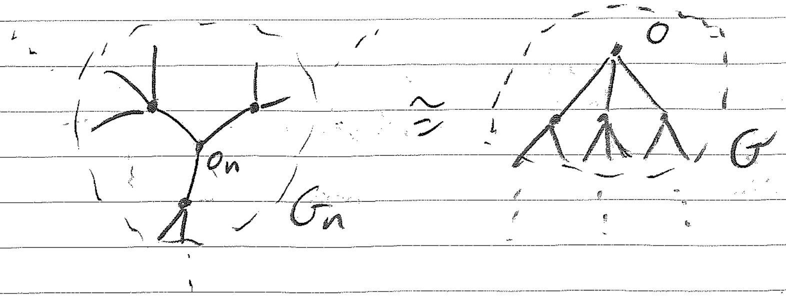
Def: Let G_n be a deterministic graph on n vertices, for each $n\ge 1$. G_n converges <u>locally weakly</u> to an infinite, connected random graph G with a distinguished root vertex o if:

Pick $o_n \in \{1,\ldots,n\}$ uniformly at random. Let $B_\ell(G_n, o_n)$ be the radius-ℓ ball around o_n in G_n, and $B_\ell(G,o)$ be that around o in G. Then for each fixed $\ell \ge 1$ and fixed rooted graph (T,o),

$$\mathbb{P}_{o_n}[B_\ell(G_n,o_n) \cong (T,o)] \to \mathbb{P}[B_\ell(G,o) \cong (T,o)]$$

as $n\to\infty$. We write $G_n \xrightarrow{wl} (G,o)$.

Note: Here $\cong$ denotes a rooted graph isomorphism, i.e. $(G_1,o_1)\cong(G_2,o_2)$ if there is a graph isomorphism mapping G_1 to G_2 and o_1 to o_2. The probability on the left is over the random choice of root o_n in G_n, and on the right over the distribution of (G,o).



Example: Let $G_n \sim G(n, \frac{d}{n})$ (Erdos-Renyi). Consider first

$$\mathbb{E}\left[\mathbb{P}_{o_n}\left[B_\ell(G_n, o_n) \simeq (T, o) \right]\right],$$ where outer $\mathbb{E}$ is over G_n.

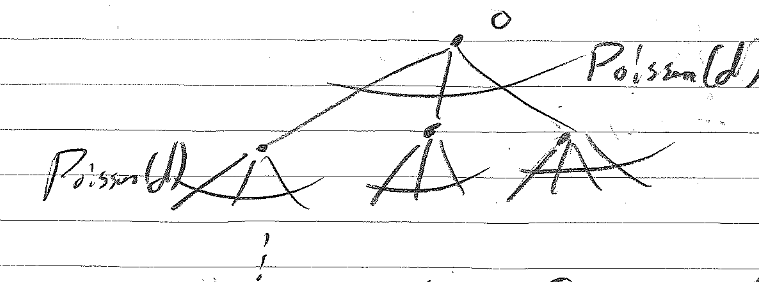
This is $\mathbb{E}\left[\frac{1}{n} \sum_{o_n=1}^{n} \mathbb{I}\{ B_\ell(G_n, o_n) \simeq (T, o)\}\right]$

$$= \mathbb{P}\left[B_\ell(G_n, 1) \simeq (T, o)\right],$$ where now $\mathbb{P}$ is over G_n.

Note that:

- Vertex 1 has degree $D_1 \sim \mathrm{Binom}(n-1, \frac{d}{n}) \xrightarrow{L} \mathrm{Poisson}(d)$.
- Conditional on the D_1 neighbors of vertex 1,
  $$\mathbb{P}[\text{no two nbrs of 1 are connected}] \simeq \left(1 - \frac{d}{n}\right)^{\binom{D_1}{2}} \to 1.$$
- Conditional on the D_1 neighbors of vertex 1, and the event that no two nbrs of 1 are connected, the first neighbor of 1 has $D_2 \sim \mathrm{Binom}(n-1-D_1, \frac{d}{n}) \xrightarrow{L} \mathrm{Poisson}(d)$ additional neighbors.
- Conditional on these D_2 neighbors,
  $$\mathbb{P}[\text{no two of these } D_2 \text{ nbrs are connected}] = \left(1 - \frac{d}{n}\right)^{\binom{D_2}{2}} \to 1.$$
- Etc...

So the law of $B_\ell(G_n, 1)$ converges to an ℓ-layer <u>Galton-Watson tree with degree distribution Poisson(d)</u>:



(Each Poisson d variable is independent.)

Letting (G, o) be the law of this infinite tree, for any fixed ℓ, T

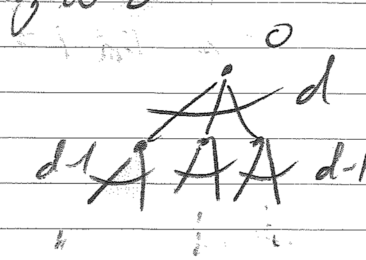
$$\mathbb{E}\left[\mathbb{P}_{o_n}\left[B_\ell(G_n, o_n) \simeq (T, o)\right]\right] \to \mathbb{P}\left[B_\ell(G, o) \simeq (T, o)\right].$$

In fact, can show

$$\mathbb{P}_{o_n}\left[B_\ell(G_n, o_n) \simeq (T, o)\right] \to \mathbb{P}\left[B_\ell(G, o) \simeq (T, o)\right]$$

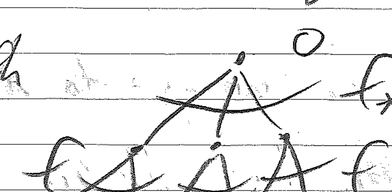
almost surely, and hence $G_n \xrightarrow{wl} (G, o)$ a.s. as $n \to \infty$.

Example: Let G_n be a d-regular graph chosen uniformly at random (from all d-regular graphs). Then $G_n \xrightarrow{wl} (G, o)$ a.s., the Galton-Watson tree with fixed degree d:



Note: Number of offspring of root o is different from number of offspring of other vertices.

Def: Let f_* be a probability mass function on $\{0, 1, 2, \ldots\}$ s.t. $\sum_{k \geq 1} k f_*(k) < \infty$. The <u>GW-tree with degree distribution f_*</u> is the random rooted tree (G, o) with



where $f(j-1) = j f_*(j) / \sum_{k \geq 1} k f_*(k)$.

Remark: $f_* = \text{Poisson}(d)$ is special:

$$\sum_{k\geq 1} k f_*(k) = \sum_{k\geq 1} k \cdot e^{-d} \cdot \frac{d^k}{k!} = d \cdot \sum_{k \geq 1} e^{-d} \cdot \frac{d^{k-1}}{(k-1)!} = d.$$

So $j f_*(j) / \sum_{k\geq 1} k f_*(k) = j \cdot e^{-d} \cdot \frac{d^j}{j!} / d = e^{-d} \frac{d^{j-1}}{(j-1)!}$, and $f = f_*$.

In general, $f \neq f_*$. The form of f comes from the following:

Choose o uniformly at random. Conditional on $\deg(o) \geq 1$, choose a random neighbor v of o. Then, letting $n_+ = \#$ vertices of G_n w/ degree ≥ 1

$$f(k-1) = \lim_{n\to\infty} \mathbb{P}[\deg(v) = k \mid \deg(o) \geq 1]$$

$$= \lim_{n\to\infty} \frac{1}{n_+} \sum_{o : \deg(o) \geq 1} \frac{1}{\deg(o)} \cdot \sum_{v \sim o} \mathbb{1}\{\deg(v) = k\}$$

$$= \lim_{n\to\infty} \underbrace{\frac{n}{n_+}}_{\to \frac{1}{1-f_*(0)}} \cdot \underbrace{\frac{1}{n} \sum_{v : \deg(v) = k}}_{\to f_*(k)} \underbrace{\sum_{o \sim v} \frac{1}{\deg(o)}}_{\to k \cdot \mathbb{E}_{j \sim f}\left[\frac{1}{1+j}\right],\ \text{independent of } \mathbb{1}\{\deg(v)=k\}}$$

$$\propto k \cdot f_*(k).$$

The law of (G, o) is unimodular — "G looks the same from any choice of root o."

Thm: (a) Suppose $\{\deg(o_n)\}_{n \geq 1}$ is uniformly integrable, and $G_n \xrightarrow{\text{wls}} (G, o)$ a.s. as $n \to \infty$. Then there exists an a.s. limit μ for the ESD of the adjacency matrix of G_n.

(b) Suppose, in addition, that $(G, o) \sim$ GW tree w/ degree distribution f_*, and $\sum_{k \geq 1} k^2 f_*(k) < \infty$. Let $\mathcal{H}$ be the space of analytic functions $f : \mathbb{C}^+ \to \mathbb{C}^+$ s.t. $|f(z)| \leq \frac{1}{\operatorname{Im} z}$. Then there is a unique distribution Q over $\mathcal{H}$ s.t. if $Y(z) \sim Q$, then

$$Y(z) \stackrel{L}{=} -\left(z + \sum_{i=1}^{N} Y_i(z)\right)^{-1}$$

where $N \sim f$, $Y_1, Y_2, \ldots \sim Q$ and these are independent. Furthermore, the Stieltjes transform of μ is $m(z) = \mathbb{E}[X(z)]$ where

$$X(z) \stackrel{L}{=} -\left(z + \sum_{i=1}^{N_*} Y_i(z)\right)^{-1},$$

and $N_* \sim f_*$, $Y_1, Y_2, \ldots \sim Q$ are independent.

Intuition: Let A be the adjacency matrix of the infinite graph (G, o), as an operator on $l_2(\mathbb{N})$. Let $R(z) = (A - zI)^{-1}$ be its resolvent. Then $m_n(z) \to \mathbb{E}[o^T R(z) o]$ a.s., where $o = (1, 0, 0, \ldots)$ and the first coordinate is the root o. $X(z)$ above is equal in law to $o^T R(z) o$.

Proof of Thm:

① Uniqueness of solution Q: We apply the Banach fixed point theorem:

If $(\mathcal{X}, d)$ is a complete metric space and $\Phi : \mathcal{X} \to \mathcal{X}$ a contraction ($d(\Phi(x), \Phi(y)) \leq c\, d(x,y)$ $\forall x, y \in \mathcal{X}$, and some $c \in [0,1)$) then there is a unique $x_* \in \mathcal{X}$ where $\Phi(x_*) = x_*$. Furthermore, $x_* = \lim_{k\to\infty} \Phi^k(x)$ for any $x \in \mathcal{X}$.

Let $\mathcal{X} = \mathcal{P}(\mathcal{H})$ be the space of probability distributions on $\mathcal{H}$.

Note that $\mathbb{E}[N] = \sum_{j \geq 1} j f(j) < \infty$ by the condition $\sum_{k \geq 1} k^2 f_*(k) < \infty$.

Define $\mathbb{D} = \{z \in \mathbb{C}^+ : \operatorname{Re} z \in [-1, 1],\ \operatorname{Im} z \in [\sqrt{\mathbb{E}N} + 1, \sqrt{\mathbb{E}N} + 2]\}$, and

$$d(P, Q) = \inf_{\substack{\text{couplings of} \\ P \text{ and } Q}} \mathbb{E}_{(X,Y) \sim (P,Q)} \int_{\mathbb{D}} |X(z) - Y(z)|\, dz.$$

Note: If $d(P, Q) = 0$, then $\exists$ coupling where $X(z) = Y(z)$ for a.e. $z \in \mathbb{D} \Rightarrow$

$X(z)=Y(z)\ \forall z\in\mathbb{C}^+$ by analyticity. So $P=Q$. Can check d is a complete metric on $\mathcal{P}(\mathcal{H})$.

Define $\Psi:\mathcal{P}(\mathcal{H})\to\mathcal{P}(\mathcal{H})$ s.t. $\Psi(Q)$ is the law of $-(z+\sum_{i=1}^N Y_i(z))^{-1}$, where $N\sim f$ and $Y_1,Y_2,\ldots\sim Q$. For any $P,Q\in\mathcal{P}(\mathcal{H})$, let (X_i,Y_i) have the coupled law s.t. $d(P,Q)=\mathbb{E}\int_D|X(z)-Y(z)|\,dz$. Then

$$d(\Psi(P),\Psi(Q))\le\mathbb{E}\int_D\left|-\frac{1}{z+\sum_{i=1}^N X_i(z)}+\frac{1}{z+\sum_{i=1}^N Y_i(z)}\right|dz$$

$$=\mathbb{E}\int_D\left|\frac{\sum_{i=1}^N X_i(z)-Y_i(z)}{(z+\sum_i X_i(z))(z+\sum_i Y_i(z))}\right|dz$$

$$\le\int_D\frac{1}{(\operatorname{Im}z)^2}\cdot\mathbb{E}\left[\sum_{i=1}^N|X_i(z)-Y_i(z)|\right]dz$$

$$\le\int_D\frac{\mathbb{E}[N]}{(\operatorname{Im}z)^2}\,d(P,Q)\,dz\le\frac{\mathbb{E}[N]}{\mathbb{E}[N]+1}\cdot d(P,Q).$$

So $\exists$ unique Q where $\Psi(Q)=Q$, by Banach fixed point theorem.

② Degree truncation: Fix a constant $D>0$. Let $\widetilde{G}_n$ be the pruned graph removing all edges adjacent to vertices w/ degree $\ge D$. Let $(\widetilde{G},o)$ be the rooted graph s.t.

$$\widetilde{G}=\begin{cases}\text{single vertex } o & \text{if } \deg(o)\ge D \text{ in } G\\ \text{pruned version of } G & \text{otherwise.}\end{cases}$$

Then $\widetilde{G}_n\xrightarrow{we}(\widetilde{G},o)$. Furthermore, in (b), $(\widetilde{G},o)$ is a GW-tree w/ some offspring distribution $\widetilde{f}_*$ for the root and $\widetilde{f}$ for other vertices. By previous remark, $\widetilde{f}(j-1)=j\,\widetilde{f}_*(j)/\sum_{k\ge1}k\widetilde{f}_*(k)$.

Suppose ESD of $\widetilde{A}_n\xrightarrow{L}\widetilde{\mu}(D)$ a.s. $\widetilde{A}_n$ differs from A_n in at most $\sum_{i=1}^n\mathbb{1}\{\deg(i)\ge D\}(\deg(i)+1)$ rows, and

$\frac{1}{n}\sum_{i=1}^n\mathbb{1}\{\deg(i)\ge D\}(\deg(i)+1)\longrightarrow\mathbb{E}[\mathbb{1}\{\deg(o)\ge D\}(\deg(o)+1)]\equiv\varepsilon_D$ a.s. by $G_n\xrightarrow{wl}(G,o)$ and the given uniform integrability. We have

Lévy-distance (ESD of $\widetilde{A}_n$, ESD of A_n) $\le\frac{1}{n}\operatorname{rank}(A_n-\widetilde{A}_n)$, so

$\limsup_{n\to\infty}$ Lévy-distance (ESD of A_n, $\widetilde{\mu}(D)$) $\le\varepsilon_D$.

Since $\varepsilon_D\to0$ as $D\to\infty$, this implies $\widetilde{\mu}(D)\xrightarrow{L}\widetilde{\mu}$ as $D\to\infty$ for some $\widetilde{\mu}$, and ESD of $A_n\xrightarrow{L}\widetilde{\mu}$ a.s.

Furthermore, if $\widetilde{X}^{(D)}(z)$ is the law of $X(z)$ corresponding to $\widetilde{f}_*$, then $\widetilde{X}^{(D)}(z)\xrightarrow{L}X(z)$: Letting $Q,\widetilde{Q}$ be fixed points of Ψ corresponding to $N\sim f$ and $\widetilde{N}\sim\widetilde{f}$, and optimally coupling $(N,\widetilde{N})$ and $(X_i,Y_i)\sim(Q,\widetilde{Q})$ as before,

$$d(Q,\widetilde{Q})\le\mathbb{E}\int_D\mathbb{1}\{N\ne\widetilde{N}\}\cdot\left|-\frac{1}{z+\sum_{i=1}^N X_i(z)}+\frac{1}{z+\sum_{i=1}^{\widetilde{N}} Y_i(z)}\right|dz$$

$$+\mathbb{E}\int_D\mathbb{1}\{N=\widetilde{N}\}\left|-\frac{1}{z+\sum_{i=1}^N X_i(z)}+\frac{1}{z+\sum_{i=1}^N Y_i(z)}\right|dz$$

$$\le\frac{C}{\operatorname{Im}z}\cdot\mathbb{P}[N\ne\widetilde{N}]+\frac{1}{(\operatorname{Im}z)^2}\mathbb{E}[N\cdot\mathbb{1}\{N=\widetilde{N}\}]\cdot d(Q,\widetilde{Q}).$$

So $d(Q,\widetilde{Q})\le C\cdot\mathbb{P}[N\ne\widetilde{N}]$ for fixed z. Similarly, letting $P,\widetilde{P}$ be laws of $X(z)$ and $\widetilde{X}(z)$, $d(P,\widetilde{P})\le C\cdot\mathbb{P}[N_*\ne\widetilde{N}_*]+C'\cdot d(Q,\widetilde{Q})$.

So as $D\to\infty$, $\mathbb{P}[N\ne\widetilde{N}]\to0$ and $\mathbb{P}[N_*\ne\widetilde{N}_*]$, implying $d(P,\widetilde{P})\to0$, i.e. $\widetilde{X}^{(D)}(z)\xrightarrow{L}X(z)$.

Thus: It suffices to show both (a) and (b) assuming G_n has max degree D.

③ Argument for bounded degree: By Skorokhod Representation, construct $\{o_n\}_{n\ge1}$ and (G,o) on a common probability space, s.t. for all fixed l, $B_l(G_n,o_n)\cong B_l(G,o)$ a.s. for all large n.

Then also $B_{l_n}(G_n,o_n) \cong B_{l_n}(G,o)$ for some sequence $l_n \to \infty$, a.s. for all large n. This implies

$$A_n e_k \to A e_k \quad \text{a.s. for each fixed } k,$$

where A is the adjacency matrix of G w/ first vertex o, and A_n is the adjacency matrix of G_n w/ first vertex o_n.

When G_n, G have bounded max degree, A_n, A are bounded self-adjoint operators on $\ell_2(\mathbb{N})$. Then also

$$(A_n - zI)^{-1} e_k \to (A - zI)^{-1} e_k \quad \text{a.s. for each fixed } k.$$

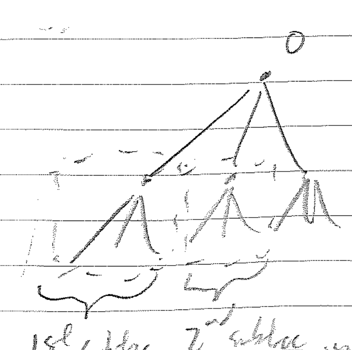
In particular, $(A_n - zI)^{-1}_{o_n o_n} \to (A - zI)^{-1}_{oo}$ a.s. Since both sides are bounded by $\frac{1}{\operatorname{Im} z}$, dominated convergence yields

$$m_n(z) = \mathbb{E}_{o_n}\left[(A_n - zI)^{-1}_{o_n o_n}\right] \to \mathbb{E}\left[(A - zI)^{-1}_{oo}\right] \equiv m(z).$$

Here $m : \mathbb{C}^+ \to \mathbb{C}^+$ and $|m(z)| \le \frac{1}{\operatorname{Im} z}$, so m is the Stieltjes transform of a measure μ. This shows (a).

For (b), consider (G,o) truncated to level L from o: Call this $(G^{(L)},o)$, and the associated adjacency matrix $A^{(L)}$. Write

$$A^{(L)} - zI = \begin{pmatrix} -z & 1\,0\cdots0 & 1\,0\cdots0 & \cdots \\ 1 & & & \\ 0 & A_1^{(L)} - zI & 0 & \\ \vdots & & & \\ 0 & & & \\ 1 & & & \\ 0 & 0 & A_2^{(L)} - zI & \\ \vdots & & & \ddots \\ 0 & \underbrace{\qquad}_{\text{first subtree}} & \underbrace{\qquad}_{\text{second subtree}} & \cdots \end{pmatrix}$$



Then by Schur complement,

$$(A^{(L)} - zI)^{-1}_{oo} = \frac{1}{-z - \sum_{i=1}^{N_*} (A_i^{(L)} - zI)^{-1}_{o_i o_i}}$$

where o_i is the root of the i^{th} subtree, and $N_* = \deg(o)$. Similarly,

$$(A_i^{(L)} - zI)^{-1} = \frac{1}{-z - \sum_{j=1}^{N_i} (A_{ij}^{(L)} - zI)^{-1}_{o_{ij} o_{ij}}}$$

where o_{ij} is the root of the $(ij)^{th}$ subtree, and $N_i = \#$ offspring of o_i.

Apply this recursively: Let $Q_0 \in \mathcal{P}(\mathbb{H})$ be the (deterministic) law of $Y(z) \equiv -\frac{1}{z}$.

Then $(A_i^{(L)} - zI)^{-1} \overset{\mathcal{L}}{=} \Psi^{L-2}(Q_0)$. As $L \to \infty$, this converges in law to the fixed-point Q of Ψ. Then $(A^{(L)} - zI)^{-1}_{oo} \xrightarrow{\mathcal{L}} X(z)$. By same argument as in part (a), $(A^{(L)} - zI)^{-1}_{oo} \xrightarrow{a.s.} (A - zI)^{-1}_{oo}$, since $A^{(L)}, A$ are bounded self-adjoint operators. So $(A - zI)^{-1}_{oo} \overset{\mathcal{L}}{=} X(z)$, and $m(z) = \mathbb{E}[X(z)]$.

Example: Let G_n be a uniform d-regular graph. Then $G_n \xrightarrow{lwc} (G,o)$, where $f_*(d) = 1$. The fixed-point equation is

$$Y(z) \overset{\mathcal{L}}{=} -\left(z + \sum_{i=1}^{d-1} Y_i(z)\right)^{-1}.$$

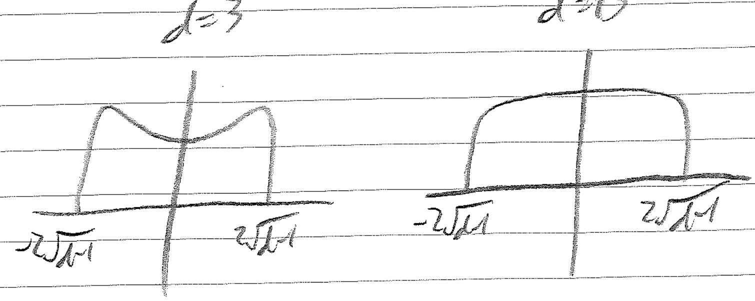
This has a <u>deterministic</u> solution solving $Y(z) = -(z + (d-1)Y(z))^{-1}$:

In fact $Y(z)$ is the Stieltjes transform of the semi-circle law w/ support $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. Then $X(z)$ is also deterministic, given by

$$X(z) = -\frac{1}{z + d\,Y(z)} = -\frac{2(d-1)}{(d-2)z + d\sqrt{z^2 - 4(d-1)}}.$$

Then $\lim_{\eta \to 0} \frac{1}{\pi} \operatorname{Im} X(\lambda + i\eta) = \frac{d}{2\pi} \cdot \frac{\sqrt{4(d-1) - \lambda^2}}{d^2 - \lambda^2}$ is the limiting density.

This is called the <u>Kesten-McKay law</u>.



As $d \to \infty$, this approaches the semi-circle law.

ple: $G_n \sim G(n, \frac{d}{n})$ Erdős-Rényi. Here, $f_\infty = f = \text{Poisson}(d)$.

The fixed point $Y(z)$ and corresponding $X(z)$ <u>are</u> random. This corresponds to individual resolvent entries $R(z)_{ii}$ being random in the large-n limit. There is no simple form for the law of $Y(z)$.

It is possible to write a fixed-point equation for the moment-generating function $f(u,z) = \mathbb{E}[e^{uY(z)}]$:

$$f(u,z) = 1 - \sqrt{u}\int_0^\infty \frac{J_1(2\sqrt{ut})}{\sqrt{t}} \cdot \exp\left(itz + d\left(f(t,z) - 1\right)\right) dt,$$

for $J_1(t) = \frac{t}{2}\sum_{k=0}^{\infty} \frac{(-t^2/4)^k}{k!(k+1)!}$ the Bessel function of the first kind.