Assignment 4 Solutions – M. Lacey, 02/18/02

Chapter 8, Problem 5 (part (c))
The asymptotic variance of the maximum likelihood estimate \( \hat{p} \) is \( \frac{1}{n I(p)} \), where
\[
I(p) = E \left[ \frac{\partial}{\partial p} \log f(X|p) \right]^2 - E \left[ \frac{\partial^2}{\partial p^2} \log f(X|p) \right].
\]
From part (b), we know that the mle \( \hat{p} = \frac{1}{X} \). Calculations for \( I(p) \):
\[
\begin{align*}
f(X|p) &= p(1-p)^{x-1} \\
\log f(X|p) &= \log p + (x-1)\log(1-p) \\
\frac{\partial}{\partial p}\log f(X|p) &= \frac{1}{p} - \frac{x-1}{1-p} \\
\frac{\partial^2}{\partial p^2}\log f(X|p) &= -\frac{1}{p^2} - \frac{x-1}{(1-p)^2}
\end{align*}
\]
Since \( E(X) = \frac{1}{p} \), it follows that \( I(p) = \frac{1}{p^2(1-p)} \), and thus the asymptotic variance of the mle is \( \frac{p^2(1-p)}{n} \).

Chapter 8, Problem 8
Using the normal approximation for the Poisson distribution, the approximate sampling distribution of \( \hat{\lambda} \) is \( N \left( \lambda_o, \frac{\lambda_o}{n} \right) \). From Example A, we’ve given that \( \hat{\lambda} = 24.9 \) and \( n = 23 \), and using the estimate \( s_\hat{\lambda} \) the resulting approximate distribution is \( N(24.9, 1.08) \). For any value \( \delta \),
\[
P(|\lambda_o - \hat{\lambda}| > \delta) \approx P \left( \frac{|Z| > \frac{\delta}{\sqrt{1.08}}}{\sqrt{1.08}} \right) = 2 \left( 1 - P \left( Z \leq \frac{\delta}{\sqrt{1.08}} \right) \right).
\]
\[
\begin{align*}
\delta = 0.5 : & \quad P \left( |Z| > \frac{0.5}{\sqrt{1.08}} \right) = 0.6307 \\
\delta = 1.0 : & \quad P \left( |Z| > \frac{1.0}{\sqrt{1.08}} \right) = 0.3363 \\
\delta = 1.5 : & \quad P \left( |Z| > \frac{1.5}{\sqrt{1.08}} \right) = 0.1492 \\
\delta = 2.0 : & \quad P \left( |Z| > \frac{2.0}{\sqrt{1.08}} \right) = 0.0545 \\
\delta = 2.5 : & \quad P \left( |Z| > \frac{2.5}{\sqrt{1.08}} \right) = 0.0162
\end{align*}
\]

Chapter 8, Problem 11
In Example D of Section 8.4, \( f(x|\alpha) = \frac{1+\alpha x}{2}, -1 \leq x \leq 1 \) and \( -1 \leq \alpha \leq 1 \). The method of moments estimate \( \hat{\alpha} = 3X \).
(a) \( E(\hat{\alpha}) = E(3X) = 3E(X) = \frac{4}{3} \) (this can be easily verified by computing the expectation of \( X \), \( E(\hat{\alpha}) = 3\left(\frac{4}{9}\right) = \alpha \), and thus the estimate is unbiased.
(b) \( Var(\hat{\alpha}) = Var(3X) = 9Var(X) = 9\left(\frac{\alpha^2}{n}\right) \). We compute \( \sigma^2 = E(X^2) - (E(X))^2 = \frac{1}{3} - \left(\frac{4}{9}\right)^2 \), and it follows that \( Var(\hat{\alpha}) = \frac{3\alpha^2}{n} \).
(c) By the Central Limit Theorem, \( \hat{\alpha} \sim N \left( \alpha, \frac{3\alpha^2}{n} \right) \). For \( n = 25 \) and \( \alpha = 0 \), \( \hat{\alpha} \sim N(0, 3/25) \), and thus \( P(|\hat{\alpha}| > 0.5) \approx P \left( \frac{|Z| > \frac{0.5}{\sqrt{3/25}}}{\sqrt{3/25}} \right) = 2(1 - P(Z \leq 1.44)) = 0.15. \)
Chapter 8, Problem 14 (part (c))
For an i.i.d. sample of random variables with density function \( f(x|\sigma) = \frac{1}{\sigma} \exp \left( -\frac{|x|}{\sigma} \right) \), we know from part (b) that the mle \( \hat{\sigma} = \frac{1}{n} \sum |x_i| \). To find the asymptotic variance of \( \hat{\sigma} \), we need to calculate \( I(\sigma) = E \left[ \frac{\partial}{\partial \sigma} \log f(X|\sigma) \right]^2 = -E \left[ \frac{\partial^2}{\partial \sigma^2} \log f(X|\sigma) \right] \):

\[
\log f(X|\sigma) = -\log(2\sigma) - \frac{|x|}{\sigma}
\]
\[
\frac{\partial}{\partial \sigma} \log f(X|\sigma) = -\frac{1}{\sigma} + \frac{|x|}{\sigma^2}
\]
\[
\frac{\partial^2}{\partial \sigma^2} \log f(X|\sigma) = \frac{1}{\sigma^2} - \frac{2|x|}{\sigma^3}
\]
\[
-E \left[ \frac{\partial^2}{\partial \sigma^2} \log f(X|\sigma) \right] = E \left( \frac{2|x|}{\sigma^3} - \frac{1}{\sigma^2} \right)
\]

\( E(|x|) = \sigma \), so \( I(\sigma) = \frac{\sigma}{\sigma^2} \), and the asymptotic variance \( \frac{1}{nI(\sigma)} = \frac{\sigma^2}{n} \).

Chapter 8, Problem 25
Electronic components have lifetimes that are exponentially distributed with density function \( f(t|\tau) = (1/\tau) \exp(-t/\tau), t \geq 0 \). Of five new components which are tested, one fails after 100 days.
(a) Let \( T \) be the time of the first failure. The distribution of \( T \) is exponential with parameter \( \frac{n}{\tau} = \frac{5}{\tau} \) for \( n = 5 \), so \( f(t|\tau) = \frac{5}{\tau} \exp \left( -\frac{5t}{\tau} \right) \).
(b) To find the mle, compute \( \frac{\partial}{\partial \tau} \log f(t|\tau) = \frac{\partial}{\partial \tau} \left( \log(5) - \log(\tau) - \frac{5t}{\tau} \right) = - \frac{1}{\tau} + \frac{5t}{\tau^2} \Rightarrow \hat{\tau} = 5T \).
(c) To calculate the sampling distribution of the mle \( \hat{\tau} \), note that \( P(\hat{\tau} \leq t) = P(5T \leq t) = P(T \leq t/5) = 1 - \exp \left( -\left( \frac{5}{\tau} \right) \left( \frac{t}{\tau} \right) \right) = 1 - \exp(-t/\tau) \), and thus \( f(\hat{\tau}) = (1/\tau) \exp(-t/\tau) \Rightarrow \hat{\tau} \sim \exp(1/\tau) \).
(d) For an exponential random variable \( X \) with parameter \( \lambda \), \( \text{Var}(X) = 1/\lambda^2 \). Since \( \hat{\tau} \sim \exp(1/\tau) \), \( \text{SE}(\hat{\tau}) = \sqrt{\tau^2} = \tau \).

Chapter 8, Problem 44
\( X_1, \ldots, X_n \) are i.i.d random variables with density function \( f(x|\theta) = (\theta + 1)x^\theta, 0 \leq x \leq 1 \).
(a) To find the MOME, calculate \( E(X) = \int_0^1 (\theta + 1)x^{\theta + 1}dx = \frac{\theta + 2}{\theta + 2} = \bar{X} \Rightarrow \hat{\theta} = 2\frac{n-1}{n} \).
(b) The likelihood function \( f(x_1, \ldots, x_n|\theta) = (\theta + 1)^n (\prod x_i)^\theta \), the log-likelihood \( \log f(x_1, \ldots, x_n|\theta) = n\log(\theta + 1) + \theta \log(\prod x_i) = n\log(\theta + 1) + \theta \sum \log(x_i) \), and \( \frac{\partial}{\partial \theta} \log f(x_1, \ldots, x_n|\theta) = \frac{n}{\theta + 1} + \sum \log(x_i) \). It follows that the mle \( \theta = -\sum \frac{n}{\log(x_i)} - 1 \).
(c) The second derivative of the log-likelihood \( \frac{\partial^2}{\partial \theta^2} \log f(x|\theta) = \frac{\partial}{\partial \theta} \left( \frac{n}{\theta + 1} + \log(x) \right) = -\frac{1}{(\theta + 1)^2} \), and thus \( I(\theta) = -E \frac{\partial^2}{\partial \theta^2} \log f(x|\theta) = \frac{1}{(\theta + 1)^2} \). The asymptotic variance of the mle is given by \( \frac{1}{nI(\theta)} = \frac{(\theta + 1)^2}{n} \).
(d) By the factorization theorem, \( \prod x_i \) is sufficient for \( \theta \): let \( g[T(x_1, \ldots, x_n), \theta] = (\theta + 1)^n (\prod x_i)^\theta \), and let \( h(x_1, \ldots, x_n) = 1 \).