Stat 241/541 Homework 5 Solution

Section 3.1: 4

4. At this writing, 37 Presidents have died. The probability that no two people from a group of 37 (all of whom are dead) died on the same day is

\[
\frac{365 \times 364 \times \cdots \times (365 - 37 + 1)}{365^{37}} \approx 0.15 .
\]

Thus, the probability that at least two died on the same day is 0.85.

Section 3.2: 7, 18, 32

7. We have

\[
\frac{b(n, p, j)}{b(n, p, j - 1)} = \frac{\binom{n}{j} p^j q^{n-j}}{\binom{n}{j-1} p^{j-1} q^{n-j+1}}
\]

\[
= \frac{n!}{j!(n-j)!} \frac{(n-j+1)! (j-1)! p}{n! q}
\]

\[
= \frac{(n-j+1) p}{j q}, \quad \text{where } q = 1 - p
\]

but \( \frac{(n-j+1) p}{j q} \geq 1 \) if and only if \( j \leq p(n+1) \), and so \( j = \lceil p(n+1) \rceil \) gives \( b(n, p, j) \) its largest value. If \( p(n+1) \) is an integer there will be two possible values of \( j \), namely \( j = p(n+1) \) and \( j = p(n+1) - 1 \). Because when \( j = p(n+1) \), \( b(n, p, j) = b(n, p, j-1) \).

18. \( \Pr(\text{no student gets 2 or fewer correct}) = b(340, 7/128, 0) \approx 4.96 \cdot 10^{-9} \); \( \Pr(\text{no student gets 0 correct}) = b(340, 1/1024, 0) \approx 0.717 \).

So Prosser is right to expect at least one student with 2 or fewer correct, but Crowell is wrong to expect at least one student with none correct.

32. Suppose we assign \( x \) people to the east side and \( n-x \) to the west side of the mountain. Then we want to maximize the probability:

\[
\Pr(E) = \Pr(\text{find the boy})
\]

\[
= \Pr(\text{find the boy | east})p + \Pr(\text{find the boy | west})(1 - p)
\]

\[
= [1 - (1 - u)^x]p + [1 - (1 - u)^{n-x}](1 - p)
\]

(1)

Take derivative of (1) w.r.t \( x \) and set it to zero, then we can get

\[
x_{\text{critical}} = \frac{n}{2} - \frac{\ln \left( \frac{p}{1-p} \right)}{2 \ln(1-u)}
\]
If \( u = 1 \), you only need to be sure to send at least one to each side. If \( u = 0 \), it doesn’t matter what you do. If \( 0 < u < 1 \), \( x \) should be the nearest integer to \( x_{\text{critical}} \).

**Extra problem:**

Suppose one player gets 1 dollar if she wins the game and loses 1 dollar otherwise. Let \( X_i \) and \( Y_i \) be the net gain for the players in the \( i \)-th game, respectively. Then,

\[
X_i, Y_i = \begin{cases} 
1, & \text{with prob’y } \frac{1}{2} \\
-1, & \text{with prob’y } \frac{1}{2}
\end{cases}
\]

Let \( S_n = \sum_{i=1}^n X_i \) and \( T_n = \sum_{i=1}^n Y_i \) be the total money of the two players after the \( n \)-th game. Then the players will quit if and only if

\[
S_n = T_n, \quad \text{for some } n
\]

From the setting, we can see that \((S_n, T_n)\) is a 2 dimensional random walk. Harry has provided the fact that for any random walk in less than 3 dimension, it is for sure that the random walk will return to the starting point, i.e.,

\[
\Pr(S_n = T_n = 0, \text{for some } n) \equiv 1
\]

Since \( \{S_n = T_n, \text{for some } n\} \supset \{S_n = T_n = 0, \text{for some } n\} \), we have

\[
\Pr(\text{quit the game}) = \Pr(\{S_n = T_n, \text{for some } n\}) \\
\geq \Pr(\{S_n = T_n = 0, \text{for some } n\}) \\
= 1
\]

Hence, the players will surely quit.

\[\square\]