Section 5.2: 18, 28, 34

18 What we need is to find $F_W(w)$, where $W = aX + b$, because $F_Y$ and $F_Z$ can be obtained by substituting $a = 1$ and $b = 0$ into $F_W$, respectively.

$$F_W(w) = \Pr(W \leq w) = \Pr(aX + b \leq w)$$
$$= \Pr(aX \leq w - b)$$
$$= \begin{cases} 
\Pr(X \leq (w - b)/a), & a > 0; \\
\Pr(w - b \geq 0), & a = 0; \\
\Pr(X \geq (w - b)/a), & a < 0. 
\end{cases}$$
$$= \begin{cases} 
F_X\left(\frac{w-b}{a}\right), & a > 0; \\
\Pr(w \geq b), & a = 0; \\
1 - F_X\left(\frac{w-b}{a}\right), & a < 0. 
\end{cases}$$

28 Let $L$ be the length of pregnancy, then $L \sim N(270, 10^2)$, we have

$$\Pr(\text{in the country}) = 1 - \Pr(\text{out of the country})$$
$$= 1 - \Pr(240 < L < 290)$$
$$= 1 - \Pr(-3 < \frac{L - 270}{10} < 2)$$
$$= 1 - (\Phi(2) - \Phi(-3)) = 1 - (\Phi(2) - 1 + \Phi(3))$$
$$= 2 - \Phi(2) - \Phi(3) \approx 2 - 0.9772 - 0.9987$$
$$\approx 0.0241$$

34

$$\Pr(X < Y) = \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(x)g(y)dxdy = \int_{x=0}^{\infty} f(x)(1 - G(x))dx.$$ 

Thus,

$$\Pr(X < Y) = \int_{0}^{\infty} \lambda e^{-\lambda x} \cdot e^{-\mu x} dx = \frac{\lambda}{\lambda + \mu}.$$ 

Therefore, the probability that a 100 watt bulb will outlast a 60 watt bulb is

$$\frac{1/200}{1/200 + 1/100} = \frac{1}{3}.$$
Section 7.2: 2, 6, 10

2 (a) The density function of \( Z \) is given by the convolution of \( X \) and \( Y \), i.e.,

\[
 f_Z(z) = \int_{-\infty}^{+\infty} f_X(z - y) f_Y(y) \, dy
\]

For the integrand to be non-zero, we must have

\[-1 \leq z - y \leq 1 \text{ and } -1 \leq y \leq 1\]

which implies \( \max(z - 1, -1) \leq y \leq \min(z + 1, 1) \), i.e.,

For \(-2 \leq z \leq 0\), \(-1 \leq y \leq z + 1\)
For \(0 < z \leq 2\), \(z - 1 \leq y \leq 1\)

Therefore,

\[
 f_Z(z) = \begin{cases} 
 z + 1 \cdot \frac{1}{2} \cdot \frac{1}{2} dy = \frac{z + 2}{4}, & \text{if } -2 \leq z \leq 0 \\
 \int_{z-1}^{1.5} \frac{1}{2} \cdot \frac{1}{2} dy = \frac{2 - z}{4}, & \text{if } 0 < z \leq 2 
\end{cases}
\]

(b) With similar procedure as in part (a), we can get

\[
 f_Z(z) = \begin{cases} 
 z + 6 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{z + 6}{4}, & \text{if } 6 \leq z \leq 8 \\
 6 - z \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{10 - z}{4}, & \text{if } 8 < z \leq 10 
\end{cases}
\]

(c) We must have \( f_X(z - y) f_Y(y) \) to be non-zero, i.e., \( z - 1 \leq y \leq z + 1 \) and \( 3 \leq y \leq 5 \), which implies,

if \( 2 \leq z \leq 4 \), then \( 3 \leq y \leq z + 1 \)
if \( 4 < z \leq 6 \), then \( z - 1 \leq y \leq 5 \)

Thus,

\[
 f_Z(z) = \begin{cases} 
 z - 2 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{z - 2}{4}, & \text{if } 2 \leq z \leq 4 \\
 6 - z \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{6 - z}{4}, & \text{if } 4 < z \leq 6 
\end{cases}
\]

(d) From the answer of (a), (b) and (c), one can get the set of \( z \) such that the corresponding density functions are non-zero.
For this problem, one can use the proof in Harry’s notes concerning $X \sim N(0, \sigma_1^2)$, $Y \sim N(0, \sigma_2^2)$ and $X$, $Y$ independent, then the density function of $Z = X + Y$ is

$$f_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2} \int \exp \left( -\frac{y^2}{2\sigma_2^2} - \frac{(z-y)^2}{2\sigma_1^2} \right) dy$$

$$= \ldots$$

$$= \frac{\sigma_1\sigma_2}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \sigma_2^2}} \exp \left( -\frac{1}{2} \frac{z^2}{\sigma_1^2 + \sigma_2^2} \right)$$

i.e., $Z = X + Y \sim N(0, \sigma_1^2 + \sigma_2^2)$, the details for the proof can be found in Harry’s notes of lecture 25. Now, one wants to find the density function of

$$f_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2} \int \exp \left( -\frac{(y - \mu_2)^2}{2\sigma_2^2} - \frac{(z - y - \mu_1)^2}{2\sigma_1^2} \right) dy$$

Let $y' = y - \mu_2$, $z' = z - \mu_1 - \mu_2$, then

$$f_{Z'}(z') = \frac{1}{2\pi\sigma_1\sigma_2} \int \exp \left( -\frac{(y')^2}{2\sigma_2^2} - \frac{(z' - y')^2}{2\sigma_1^2} \right) dy$$

which is exactly the same as the first equation and this implies

$$Z' = Z - \mu_1 - \mu_2 \sim N(0, \sigma_1^2 + \sigma_2^2)$$

Therefore, $Z' \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

10 The c.d.f. of $M$ is

$$\Pr(M \leq m) = \Pr \left( \min(X_1, X_2, \ldots, X_n) \leq m \right) = 1 - \Pr \left( \min(X_1, X_2, \ldots, X_n) \geq m \right) = 1 - \Pr(X_1 \geq m, X_2 \geq m, \ldots, X_n \geq m) = 1 - \Pr(X_1 \geq m) \cdot \Pr(X_2 \geq m) \cdots \Pr(X_n \geq m) = 1 - e^{-\frac{m}{\mu}} \cdot e^{-\frac{m}{\mu}} \cdots e^{-\frac{m}{\mu}} = 1 - e^{-\frac{m}{n\mu}}$$

which implies $M$ is exponentially distributed with mean $\frac{\mu}{n}$. 

3